Game Theory

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Outline

(September 3, 2007)

- Introduction
- Static Games of Complete Information: Normal Form Games
- Incomplete Information and Bayesian Games
- Behavioral Game Theory and Experimental Economics
- Dynamic Games: Extensive Form Games

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- Repeated Games
- Negotiation: Non-Cooperative Approach
- Cooperative Game Theory
- Equilibrium Refinement and signaling
- Strategic Information Transmission

Bibliography

Bibliography

- Camerer (2003) : "*Behavioral Game Theory: Experiments on Strategic Interaction*"
- Gibbons (1992) : "*Game Theory for Applied Economists*"
- Myerson (1991) : "*Game Theory: Analysis of Conflict*"
- Osborne (2004) : "*An Introduction to Game Theory*"
- Osborne and Rubinstein (1994) : "*A Course in Game Theory*"

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Non-technical:

- Dixit and Nalebuff (1991) : "*Thinking Strategically*"
- Nalebuff and Brandenburger (1996) : "*Co-opetition*"

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☞ Interactive decision theory

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Game = interactive decision situation in which the payoff/utility/satisfaction of each player depends not only on his own decision but also on others' decisions

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☞ Economic, social, political, military, biological situations

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➥ Not necessarily strictly competitive, win-loose situations; zero-sum vs. non-zero-sum games . . . image ("loose-loose situation") . . .

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(3) ➨ we modify the games (rules, transfers, . . .) in order to get solutions satisfying some properties like Pareto-optimality, anonymity, . . . Contracts, full commitment

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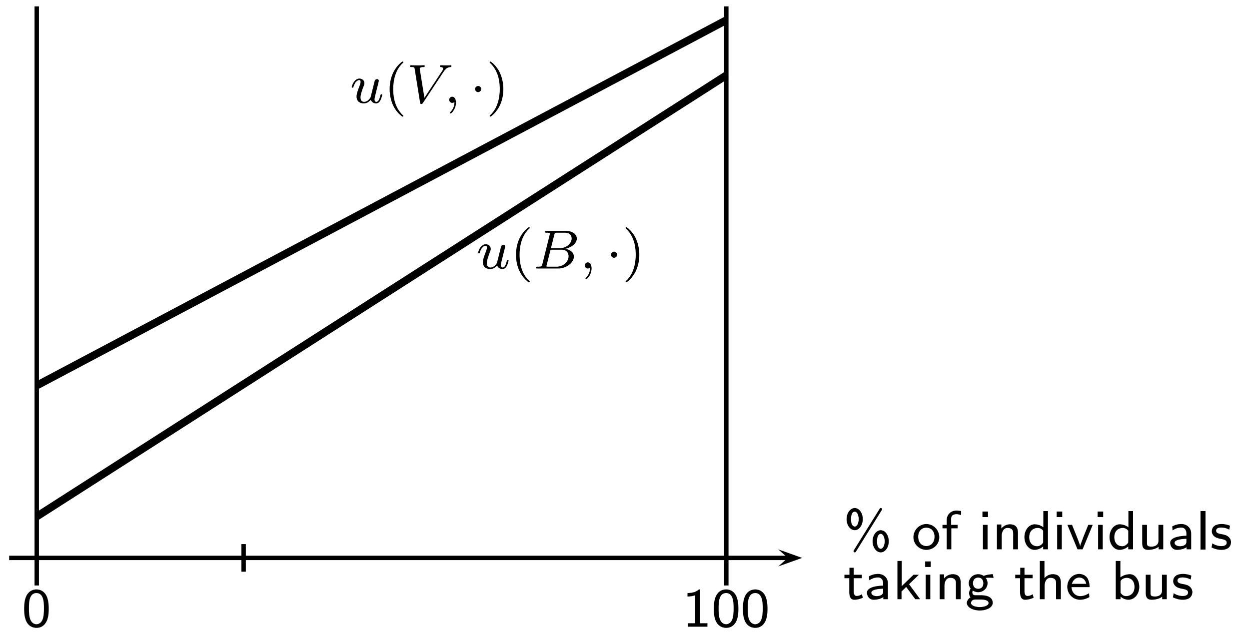

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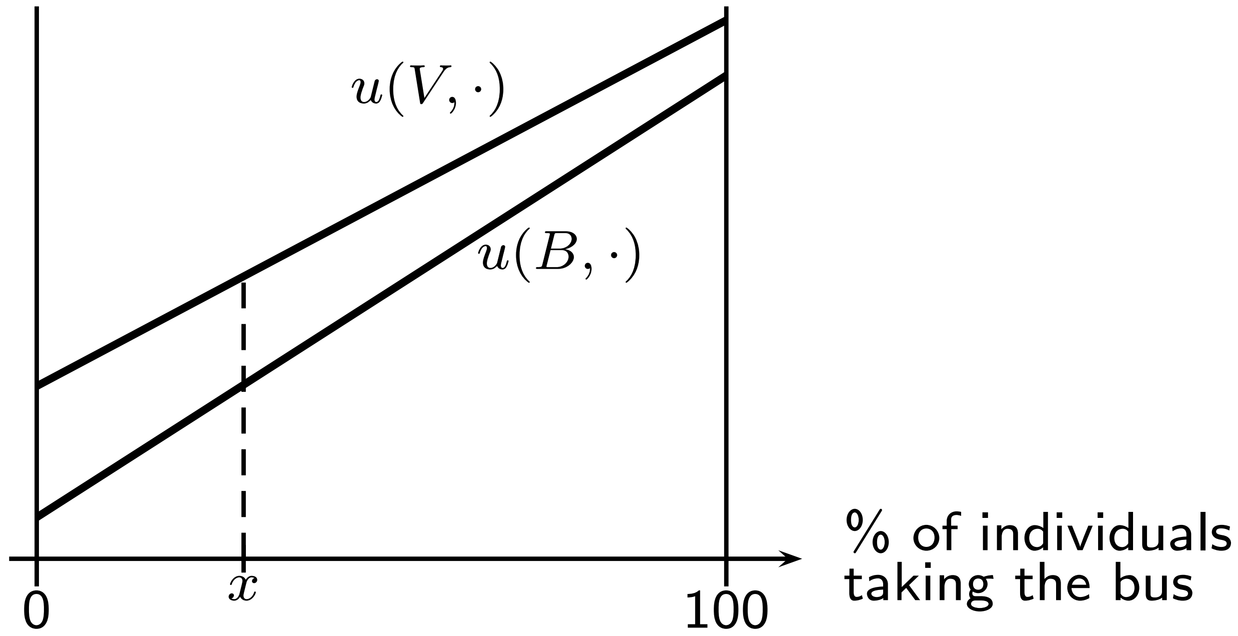

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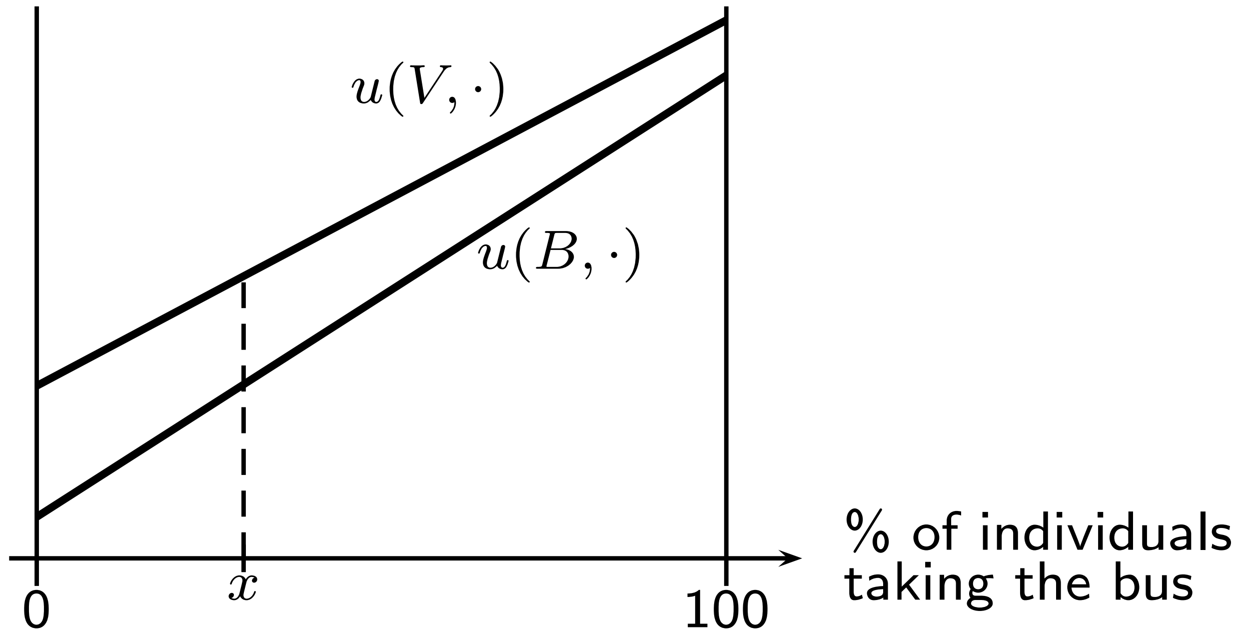

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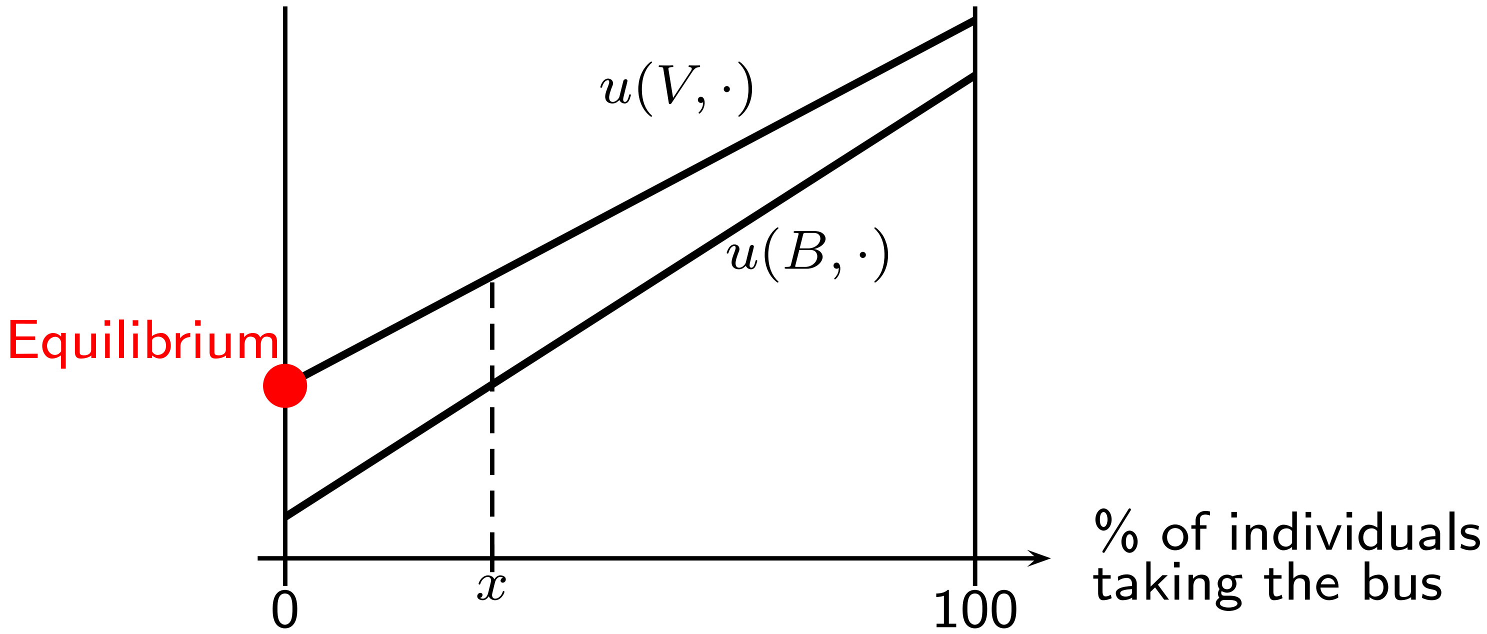

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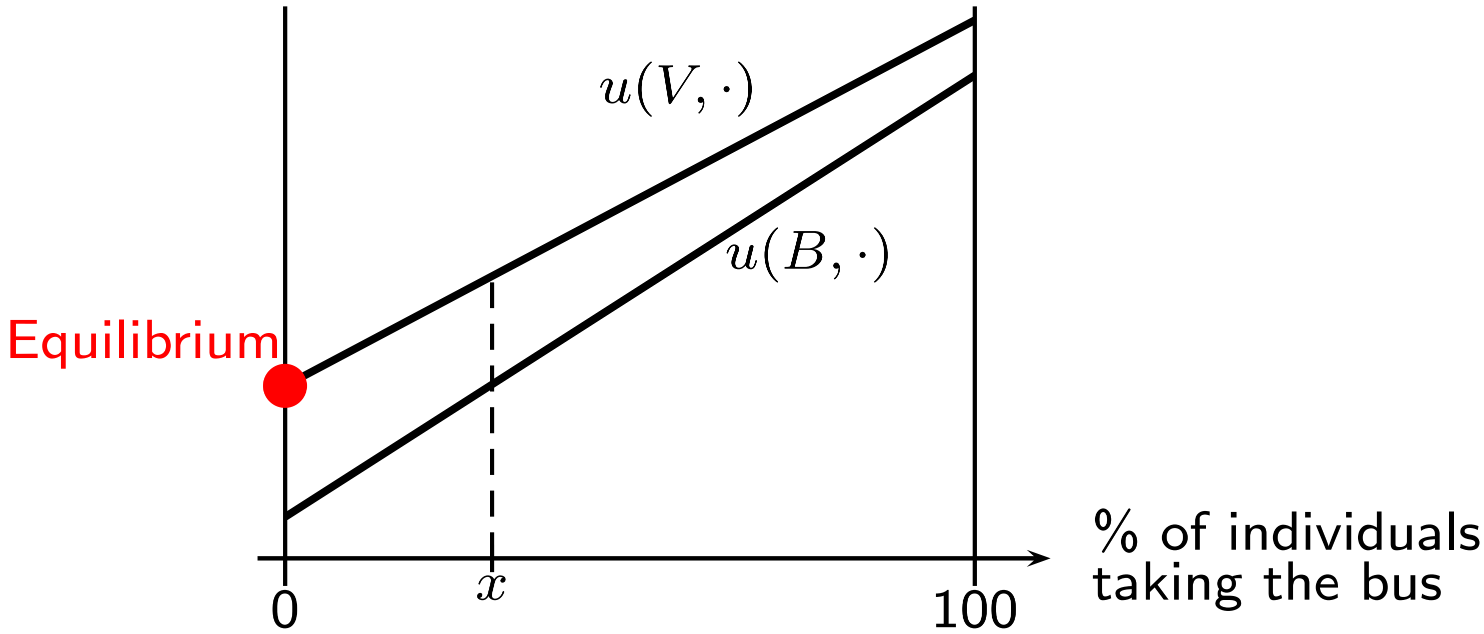

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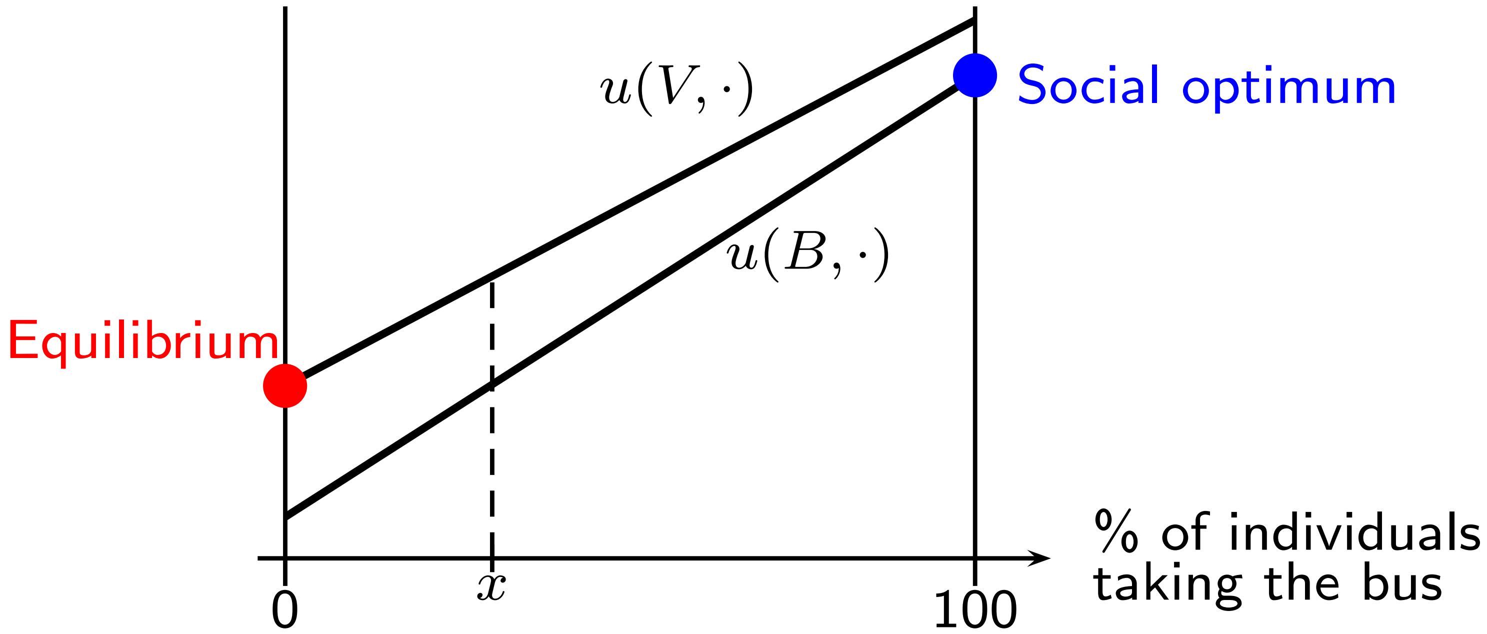

$u(V, x) > u(B, x)$ for every $x \Rightarrow$ everybody takes the car ($x = 0$)

$\Rightarrow u(V, 0)$ for everybody $\Rightarrow$ inefficient comparing to $x = 100$

New policy (taxes, toll, bus lines, . . .)

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➠ new setting

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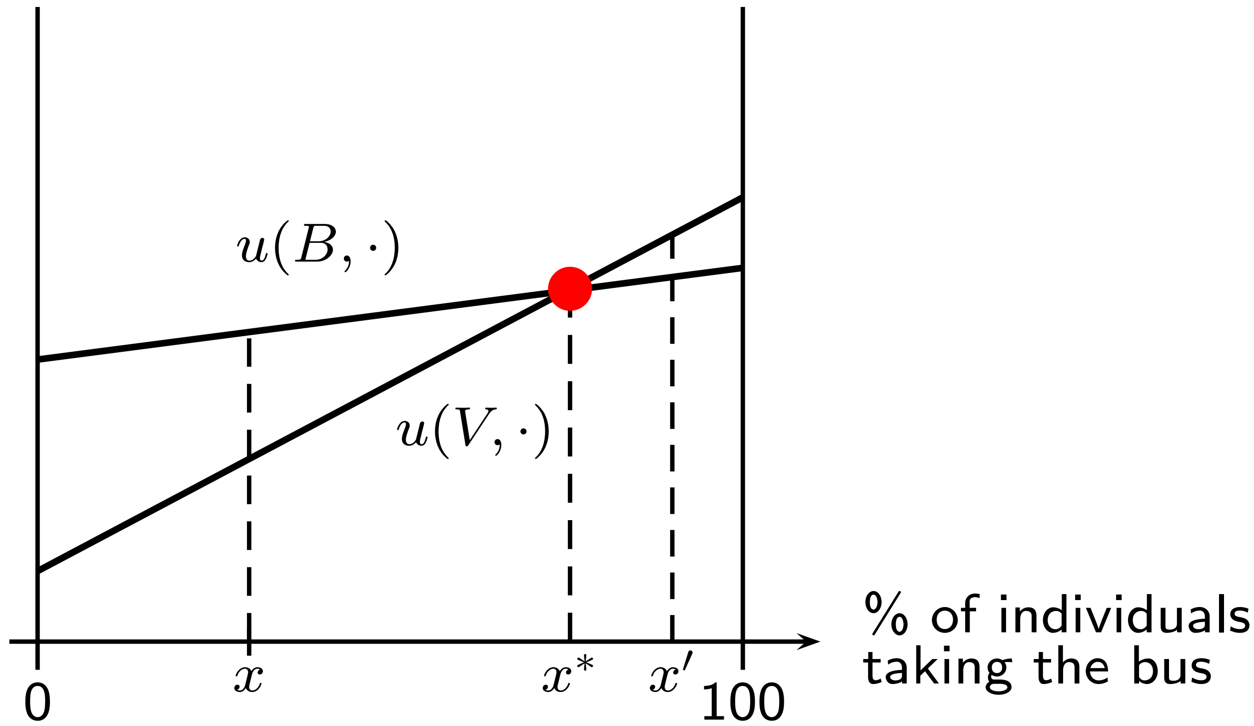

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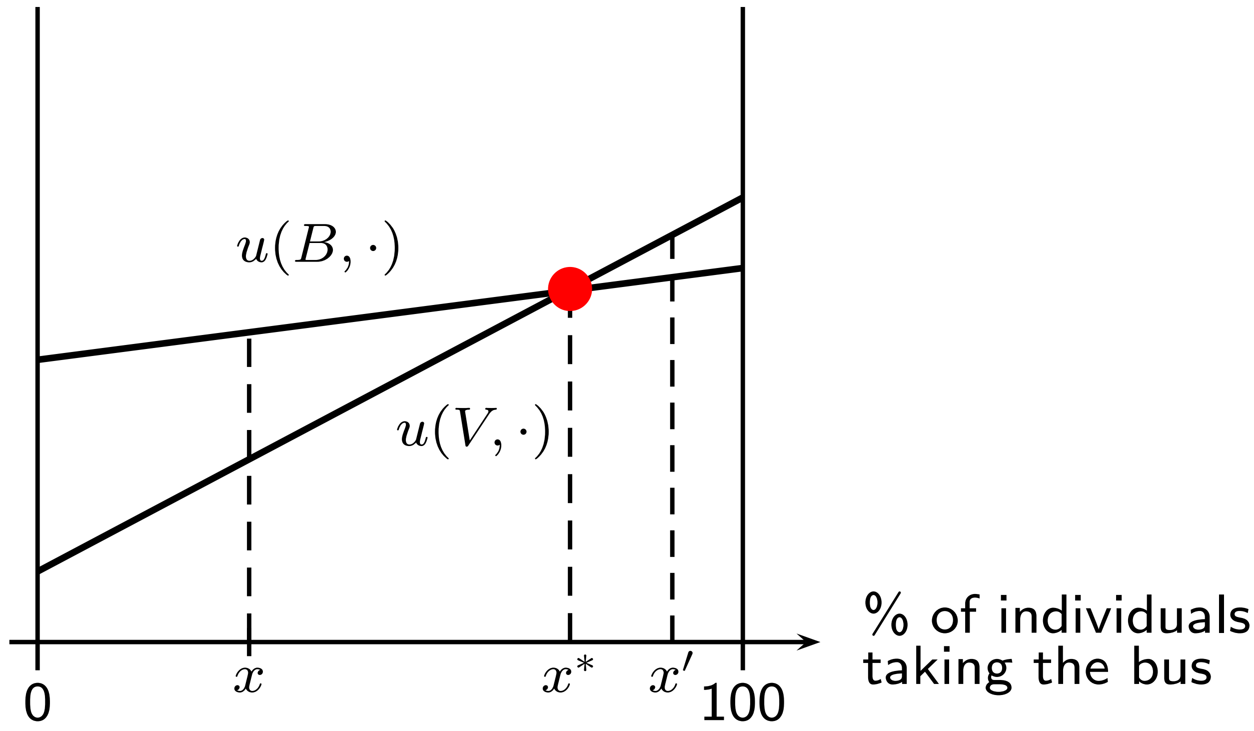

➠ New (Nash) equilibrium, more efficient (but still not Pareto optimal)

Alternative configuration: multiplicity of equilibria

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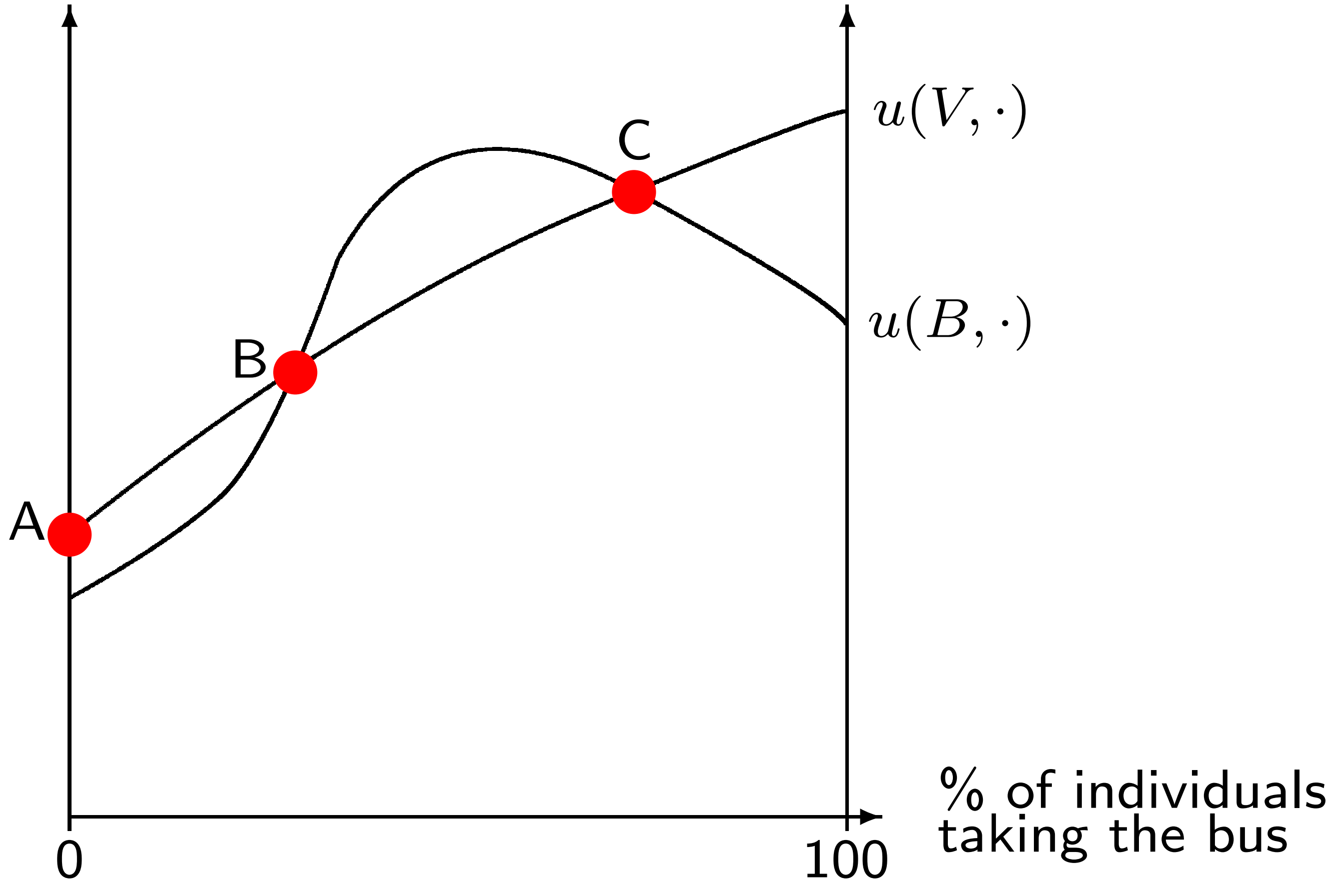

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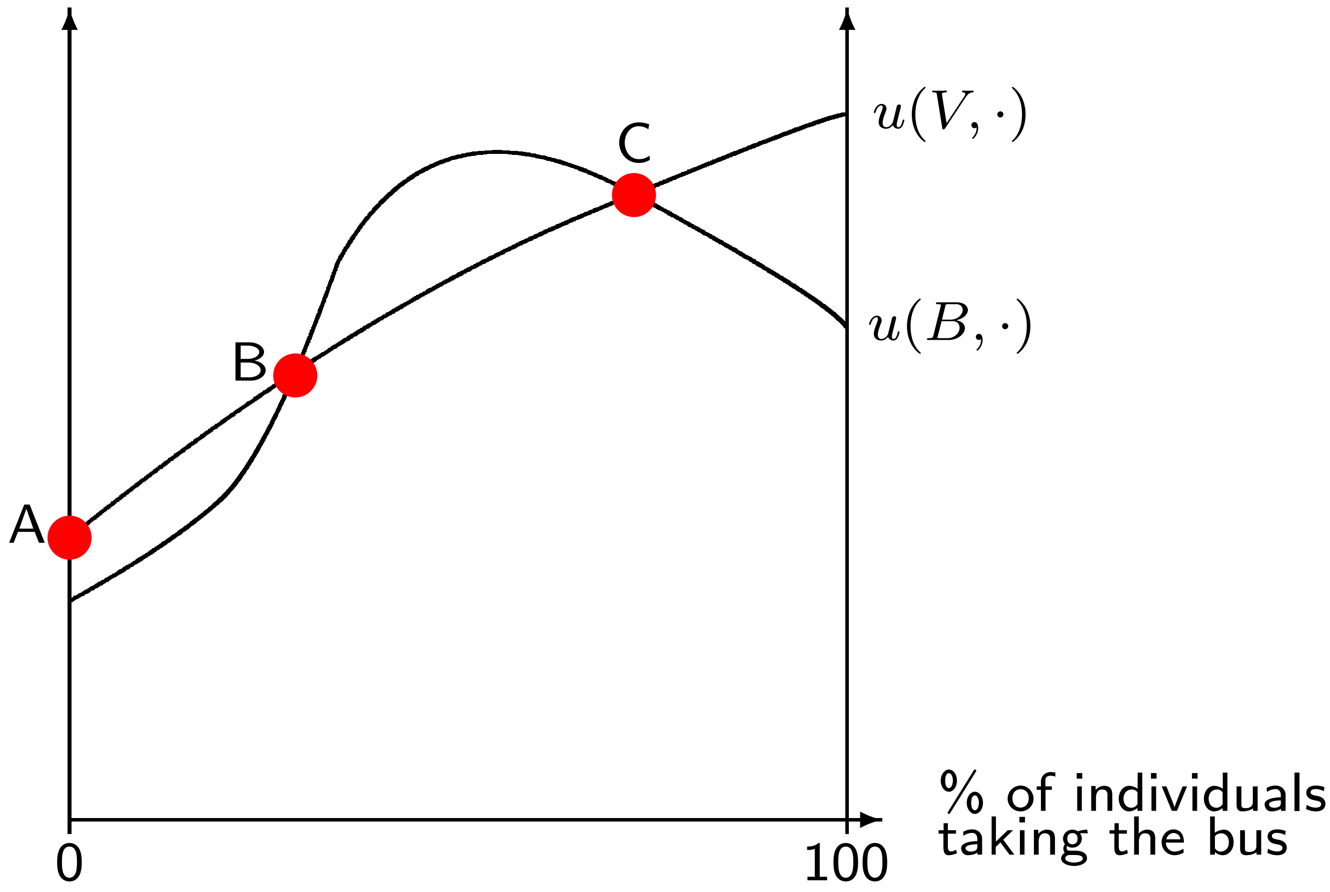
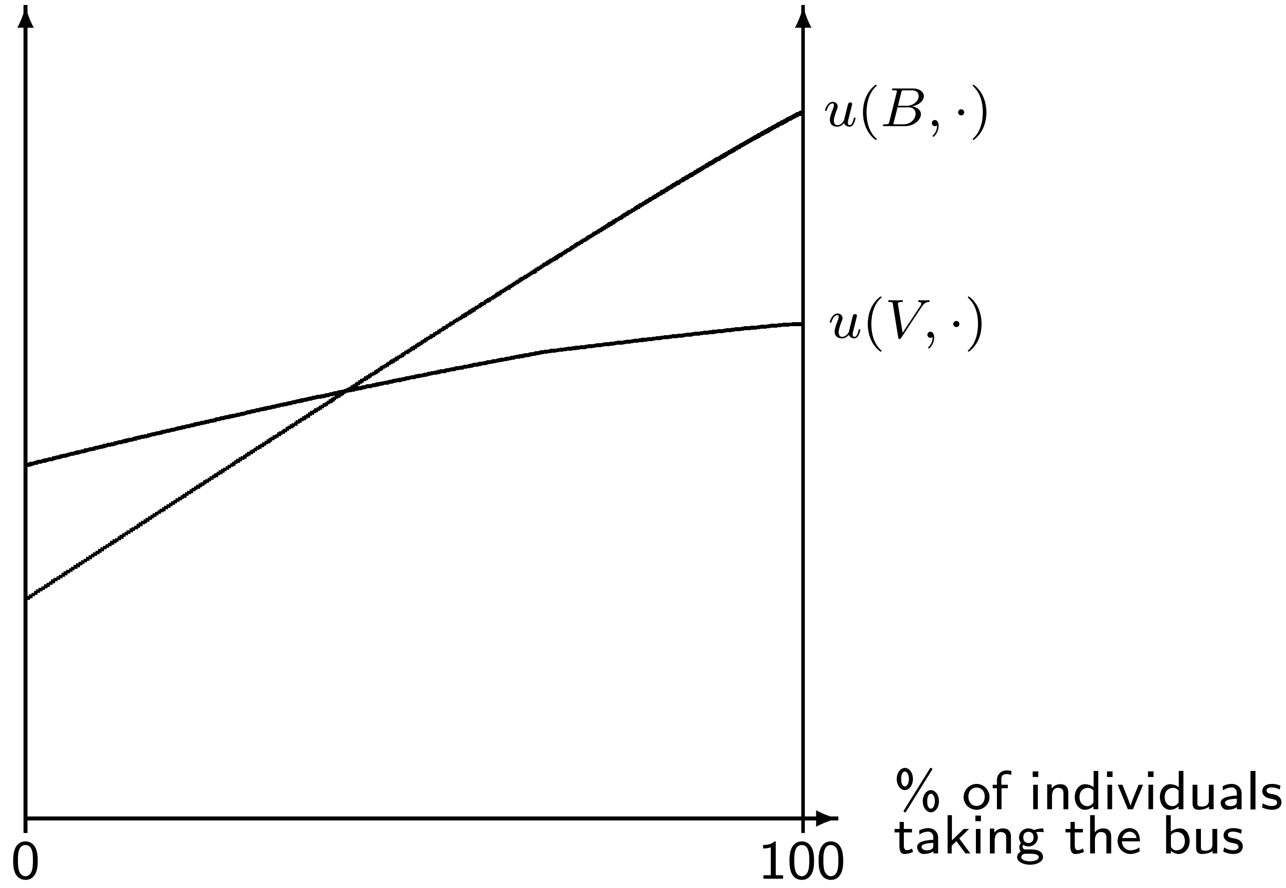


A : stable and inefficient (Pareto dominated) equilibrium

B : unstable and inefficient equilibrium

C : stable and efficient equilibrium

✍ Find the Nash equilibria in the following configuration. Which one is stable? Pareto efficient?



Second Example: Dividing the Cake

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$N = \{1, 2\}$ = two kids (**the players**)

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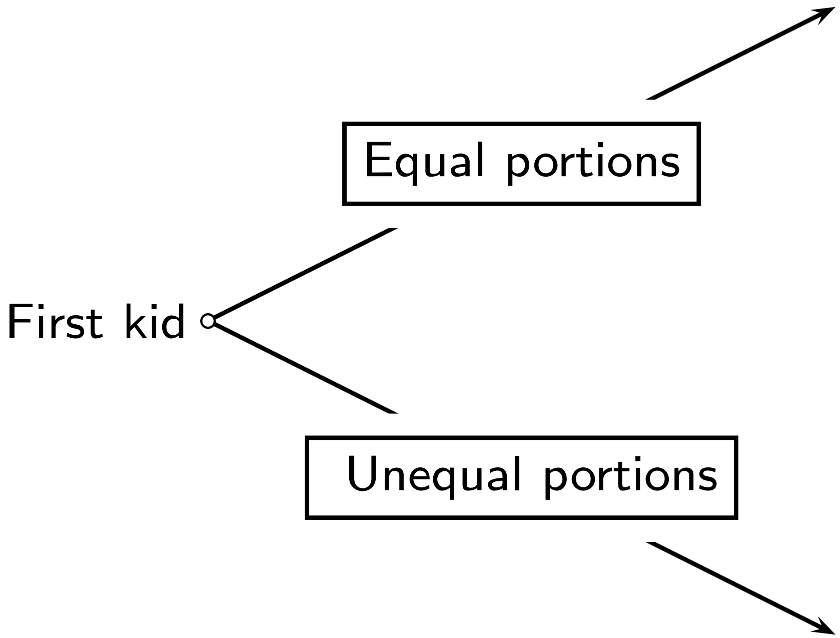
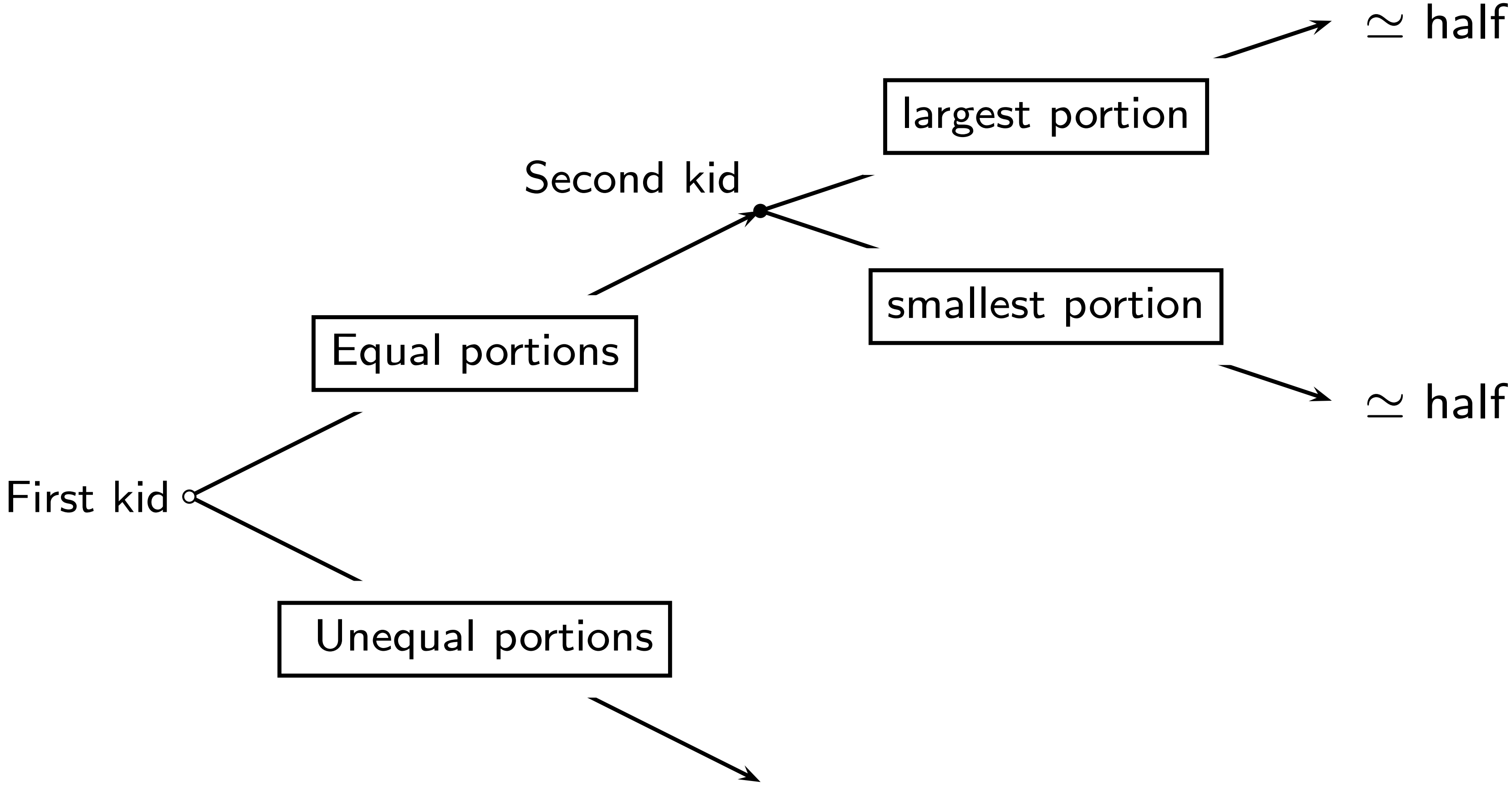
➥ Decision tree:

Extensive form game


First kid
Equal portions
Unequal portions


$\simeq$ half
largest portion
Second kid
smallest portion
Equal portions
$\simeq$ half
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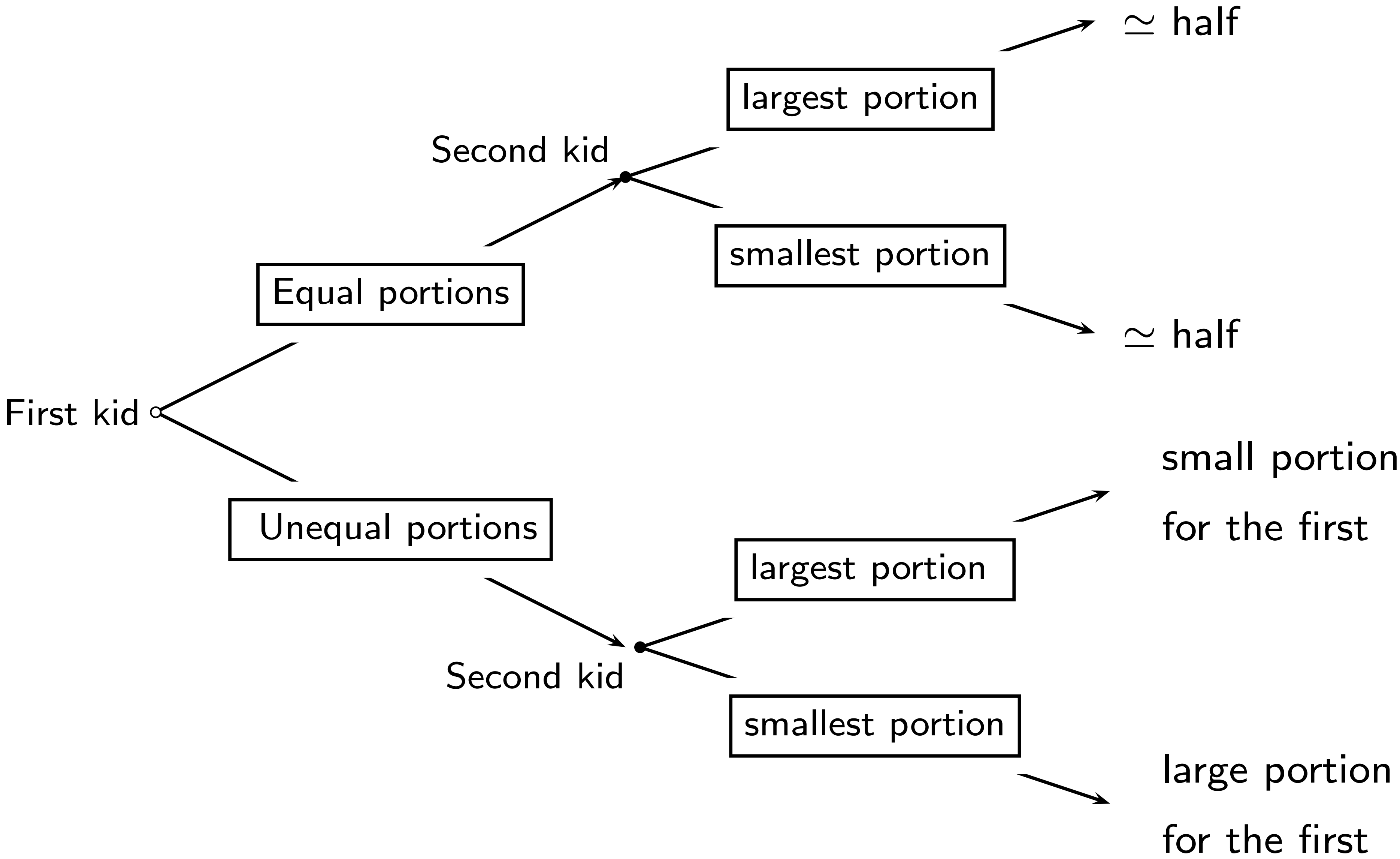
Unequal portions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First kid
small portion
for the first
Unequal portions
largest portion
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large portion
for the first

Best strategy for the first kid: divide the cake into equal portions

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➠ Fair solution, even if players are egoist, do not care about altruism or equity

Other possible representation of the game:

Table of outcomes or **Strategic/normal form game**

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2nd kid

1st kid	G	P	$(P \mid E,\ G \mid I)$	$(G \mid E,\ P \mid I)$
E	$\simeq$ half	$\simeq$ half	$\simeq$ half	$\simeq$ half
I	small portion	large portion	small portion	large portion

✍ Other simple example (except for Charlie Brown) of backward induction:

image

- Represent this situation into an extensive form game (decision tree) and find players' optimal strategies
- Represent this situation into a normal form game (table of outcomes)

Third Example: The Strategic Value of Information

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Two possible projects: a and b

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Two equally likely states/situations ($\mathrm{Pr}[\alpha] = \mathrm{Pr}[\beta] = 1/2$):

α: Only project a is profitable

β: Only project b is profitable

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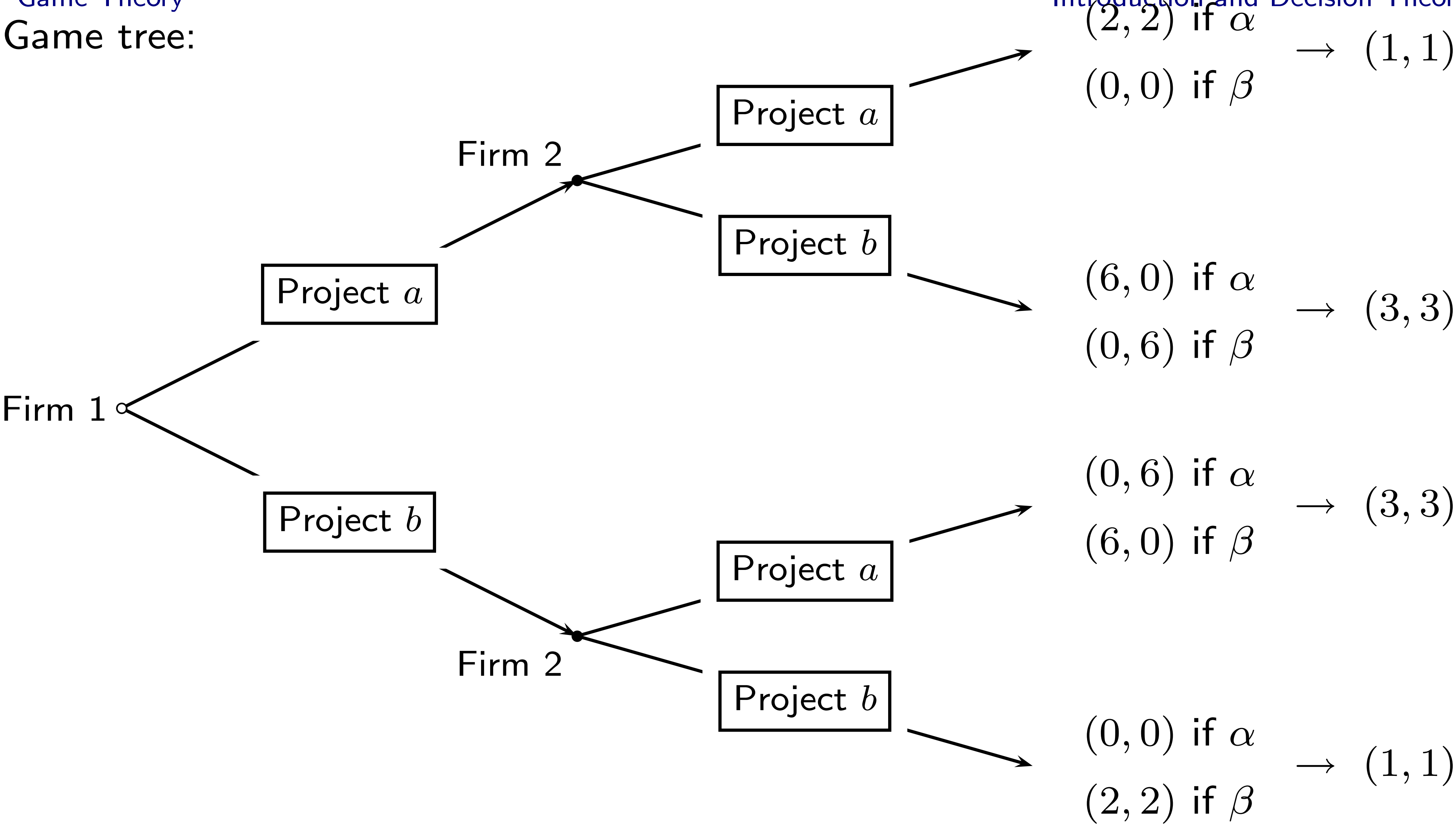
β: Only project b is profitable

- **Neither firm 1 nor firm 2 is informed.**

Game tree: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Project a
Project b
Firm 2
Project a
Project b
Firm 2
Project a
Project b
$(2,2)$ if α
$(0,0)$ if β
$\rightarrow$ $(1,1)$
$(6,0)$ if α
$(0,6)$ if β
$\rightarrow$ $(3,3)$
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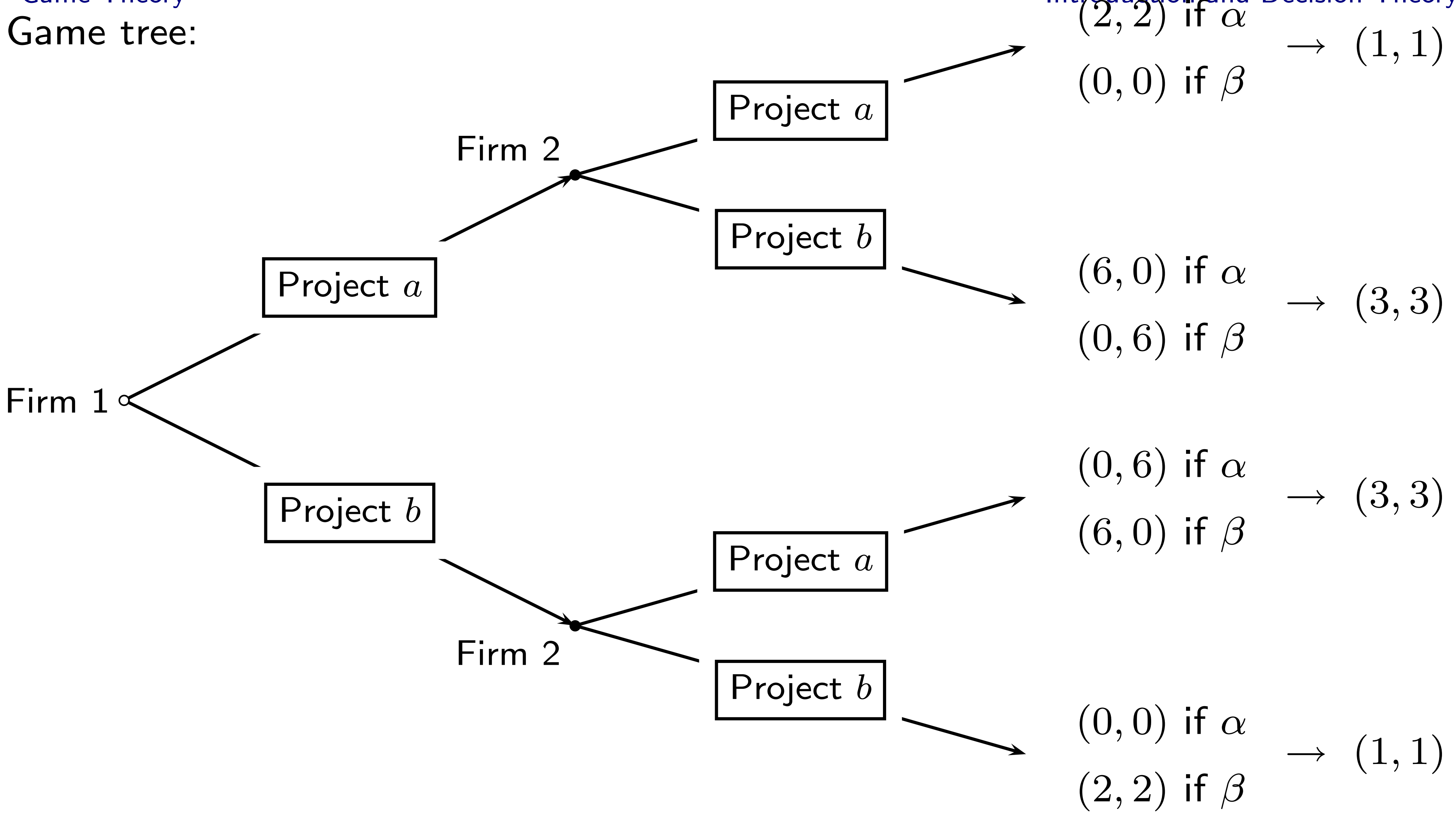

➠ Firm 2 always chooses a project different from firm 1, so each firm's expected payoff is 3

- **Firm 1 informed and Firm 2 uninformed.**

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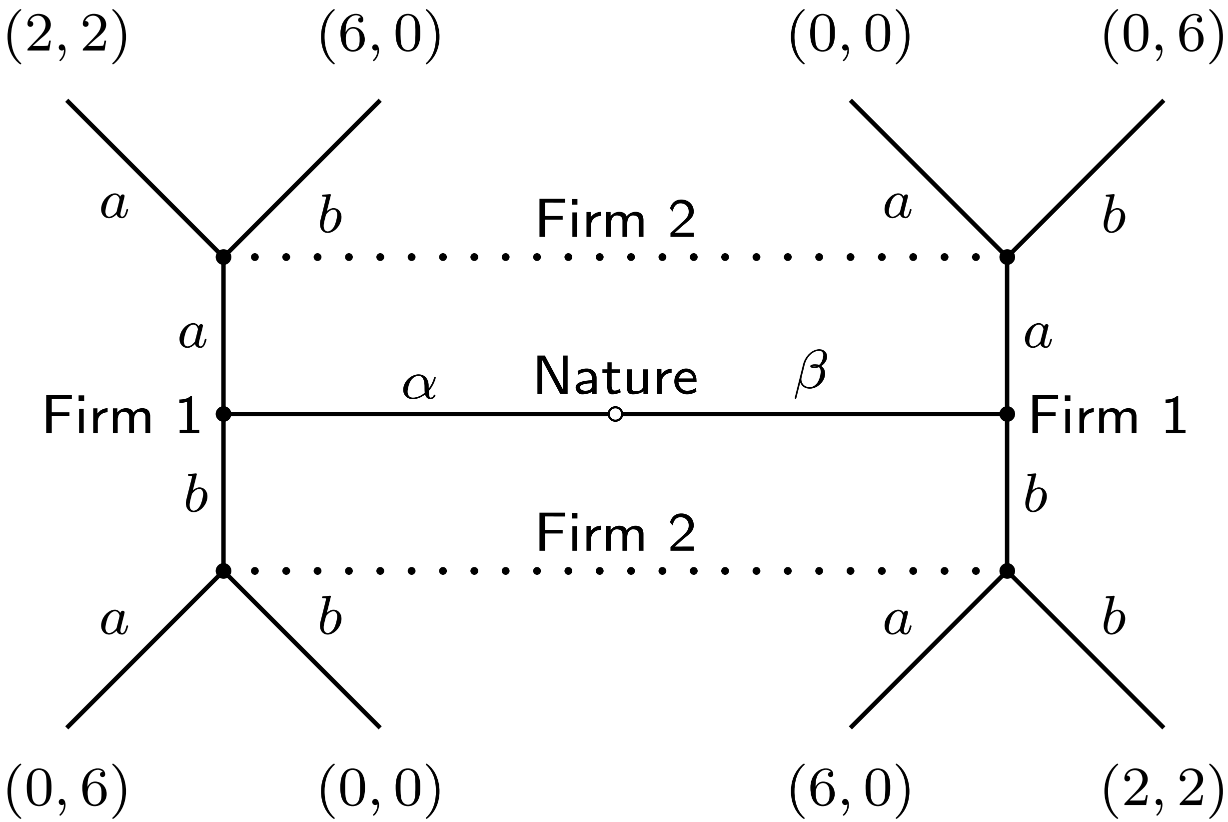
Game tree (with imperfect information):



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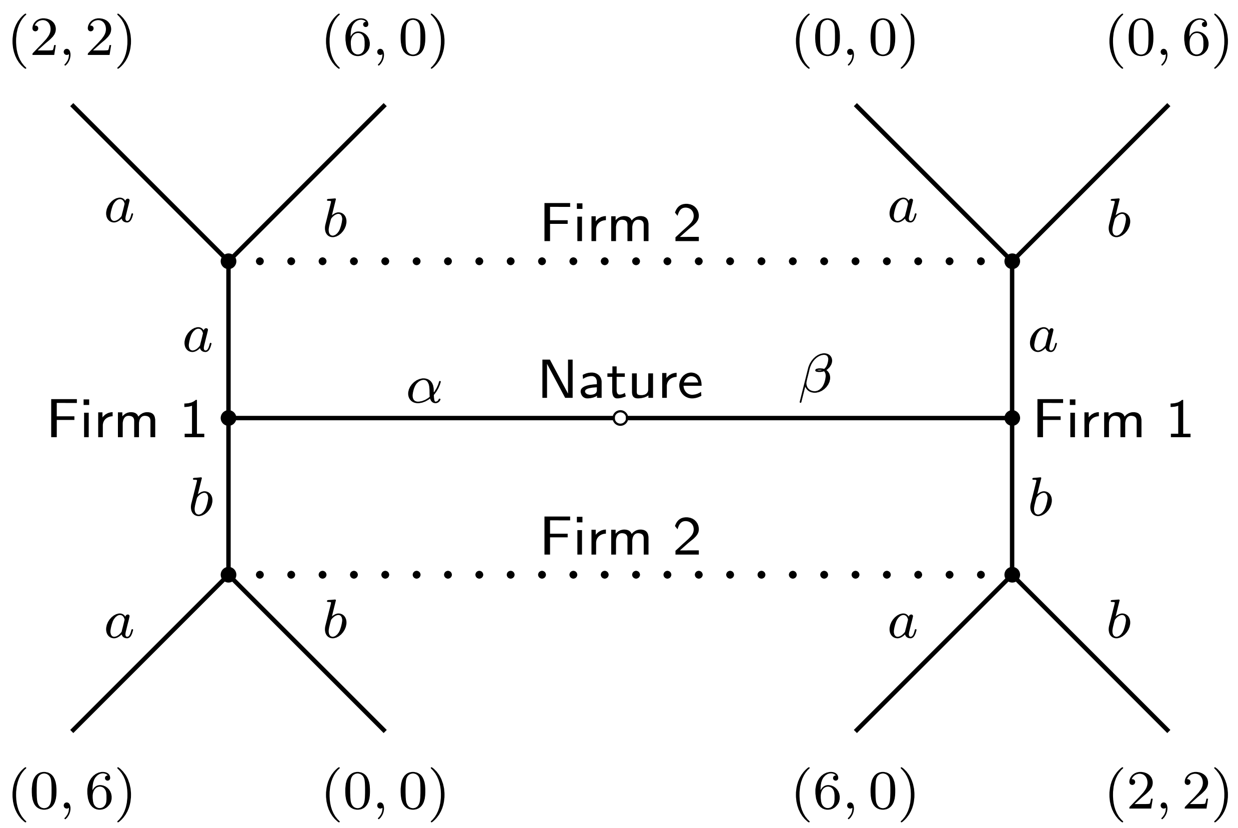

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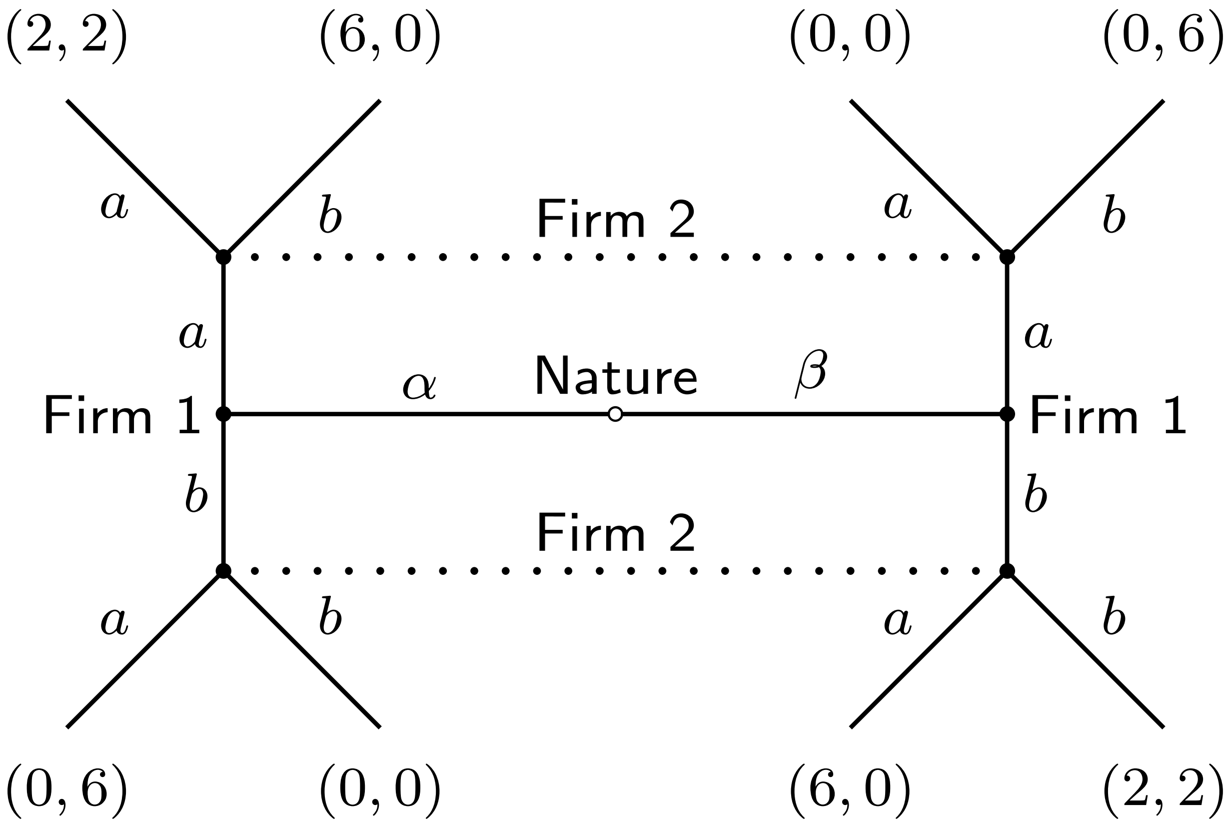

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➠ The strategic value of information is **negative** for firm 1! ($\neq$ **individual** decision problem). But Firm 2 knows that Firm 1 knows . . .

✍ Other examples:

M. Shubik (1954) “Does the fittest necessarily survive” pdf

- Understand the resolution of the game
- Do the example with other abilities
- Think about applications (e.g., elections, diplomacy, . . .)

See also “The Three-Way Duel” from Dixit and Nalebuff (1991) pdf

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- Set of **players**

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$\Rightarrow$ which solution concept is appropriate, "reasonable"?

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- Von Neumann and Morgenstern (1944), "*Theory of Games and Economic Behavior*"

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- Emile Borel (1921): mixed (random) strategy
- Von Neumann (1928): maxmin theorem (strict competition)
- Von Neumann and Morgenstern (1944), "*Theory of Games and Economic Behavior*"
- Nash (1950b, 1951): equilibrium concept, non zero-sum games

Short History

- Cournot (1838, Chap. 7): duopoly equilibrium
- Edgeworth (1881): contract curve, notion of the "core"
- Darwin (1871): evolutionary biology, natural selection
- Zermelo (1913): winning positions in chess
- Emile Borel (1921): mixed (random) strategy
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- Von Neumann and Morgenstern (1944), "*Theory of Games and Economic Behavior*"
- Nash (1950b, 1951): equilibrium concept, non zero-sum games
- Nash (1950a, 1953): bargaining solution

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- Aumann (1974, 1987): correlated equilibrium, epistemic conditions
- Lewis (1969), Aumann (1976): common knowledge

Reminder: Decision Theory

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Example of a lottery (roulette "game"):

Set of outcomes $= \{00, 0, 1, \ldots, 36\}$ (probability $1/38$ each)

Consider the two following alternatives:

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- a : Bet 10 € on even

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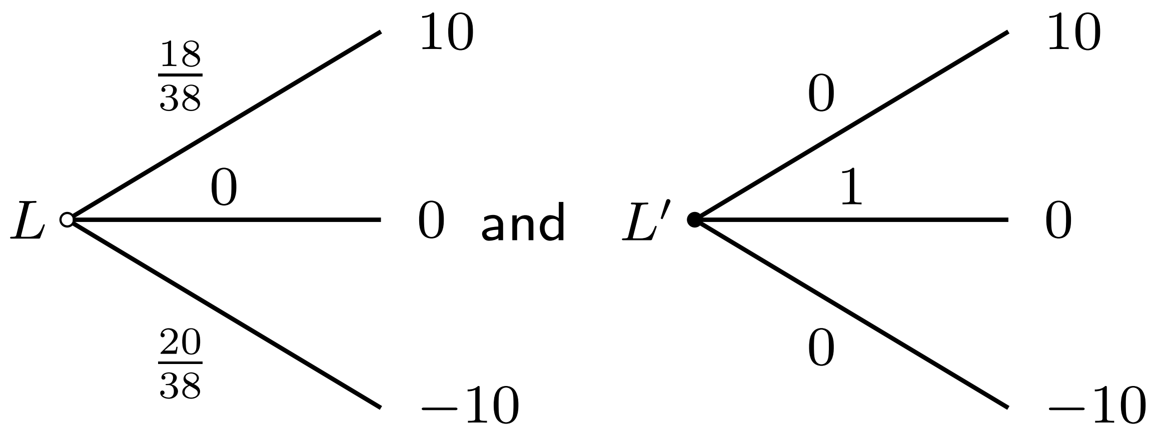

Possible decision criterion: mathematical expectation:

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$$\mathrm{E}(L) = \frac{18}{38}\,10 - \frac{20}{38}\,10 = -\frac{20}{38} \qquad \mathrm{E}(L') = 0$$

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However, most people would not pay more than 100 and even 10 euros for such a bet. . .

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For example, the expected logarithm of the payoff is:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{2^k} \ln(2^k) &= (\ln 2) \sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^k = (\ln 2) \left[2 \sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^k - \sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^k\right] \\ &= (\ln 2) \left[\sum_{k=0}^{\infty} (k+1) \left(\frac{1}{2}\right)^k - \sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^k\right] \\ &= (\ln 2) \left[1 + \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k\right] = \ln 4\end{aligned}$$

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$\Rightarrow$ Value of a certain payoff equal to 4 euros

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1944: von Neumann and Morgenstern give a rigorous axiomatics for the solution proposed by Bernoulli

Figure 1: John von Neumann (1903–1957)

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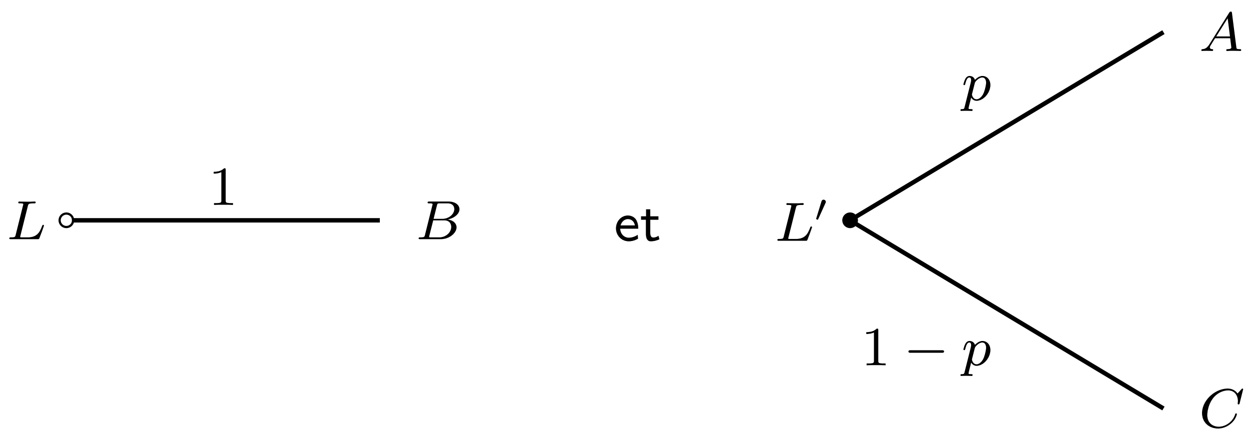
Consider the bets



and assume $L \succeq L' \Leftrightarrow p \geq 2/3$

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$$a > \frac{2}{3}a + \frac{1}{3}c > c$$

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These differences of utilities from one consequence to another one represent the individual's attitude towards risk, not a scale of satisfaction

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$$\{\alpha \in [0,1] : \alpha L + (1-\alpha)L' \succeq L''\}$$

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- Independence axiom. For all L, L', $L'' \in \mathcal{L}$ and $\alpha \in (0,1)$ we have

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Theorem of von Neumann and Morgenstern.

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If the preference relationship $\succeq$ over the set of lotteries $\mathcal{L}$ is rational, continuous and satisfies the independence axiom, then it admits an VNM expected utility representation

That is, there exist values $u(c)$ for the consequences $c \in C$ such that for all lotteries $L = (p_1, \ldots, p_C)$ and $L' = (p'_1, \ldots, p'_C)$ we have

$$L \succeq L' \Leftrightarrow \underbrace{\sum_{c \in C} p_c \, u(c)}_{U(L)} \geq \underbrace{\sum_{c \in C} p'_c \, u(c)}_{U(L')}$$

Property. (Cardinality) Let $U : \mathcal{L} \to \mathbb{R}$ be a VNM expected utility function for $\succeq$ over $\mathcal{L}$. The function $\widetilde{U} : \mathcal{L} \to \mathbb{R}$ is another VNM expected utility function for $\succeq$ if and only if there exist $\beta > 0$ and $\gamma \in \mathbb{R}$ such that

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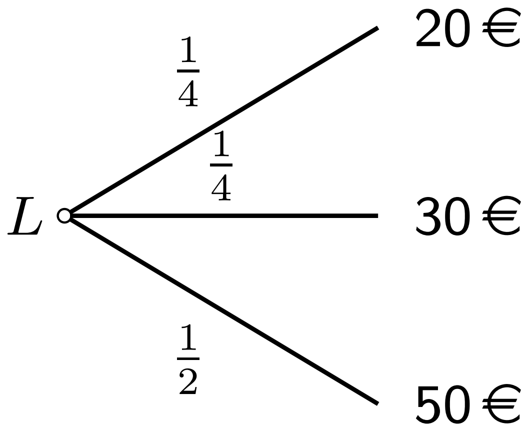
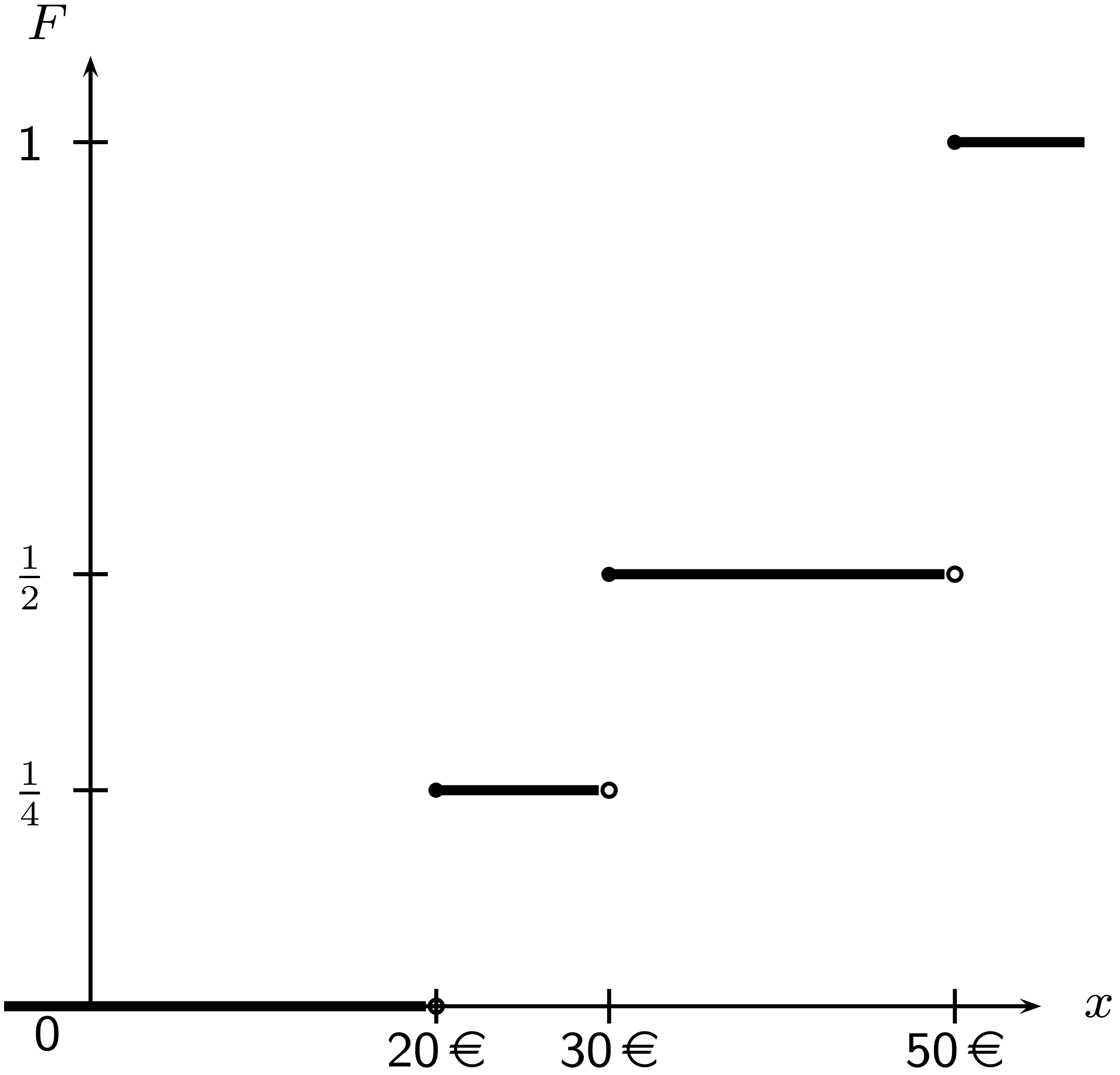
For example



$$\longrightarrow \qquad F(x) = \begin{cases} 0 & \text{if } x < 20 \\ 1/4 & \text{if } x \in [20, 30) \\ 1/2 & \text{if } x \in [30, 50) \\ 1 & \text{if } x \geq 50 \end{cases}$$


F
1
$\frac{1}{2}$
$\frac{1}{4}$
0
20 €
30 €
50 €
x

In this setting F is evaluated by the decisionmaker with

$$\begin{aligned} U(F) &= \int_C u(c)\, dF(c) \\ &= \int_C u(c) f(c)\, dc \quad \text{if the density } f \text{ exists} \end{aligned}$$

Approximation and Mean/Variance Criterion

Lottery (random variable) $\tilde{x}$

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Taylor approximation of the (Bernoulli) utility function u around $\overline{x} = E(\tilde{x})$:

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$$\Rightarrow \; U(\tilde{x}) = E[u(\tilde{x})] =$$

$$= u(\overline{x}) + \frac{u''(\overline{x})}{2!}\underbrace{E[(\tilde{x} - \overline{x})^2]}_{\sigma_x^2} + \frac{u'''(\overline{x})}{3!}E[(\tilde{x} - \overline{x})^3] + \cdots$$

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$\Rightarrow$ the expected utility of a lottery may incorporate every moment of the distribution

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- Quadratic utility function $u(x) = \alpha + \beta x + \gamma x^2 \Rightarrow$ mean/variance criterion (Markowitz, 1952)

$$U(\tilde{x}) = \alpha + \beta\overline{x} + \gamma(\overline{x}^2 + \sigma_x^2)$$

used in the CAPM "Capital Asset Pricing Model"

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- An agent is risk neutral if

$$\delta_{E(F)} \sim F \quad \forall\, F \in \mathcal{L}$$

If the preference relation $\succeq$ can be represented by an expected utility function, then the agent is risk adverse if for all lotteries F

$$u[E(F)] \equiv u\left(\int c\, dF(c)\right) \geq \int u(c)\, dF(c) \equiv U(F)$$

(Jensen inequality for concave utility functions)

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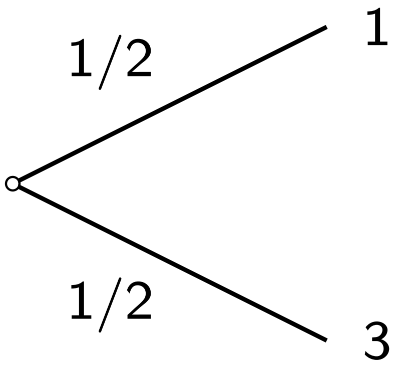
(Jensen inequality for concave utility functions)

$\Rightarrow$ An agent is (strictly) risk averse if and only if his utility function u is (strictly) concave. An agent is risk neutral if and only if his utility function u is linear

Example:

$$F = (\tfrac{1}{2}, 1; \tfrac{1}{2}, 3) =$$



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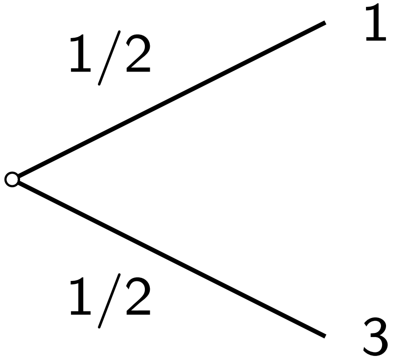

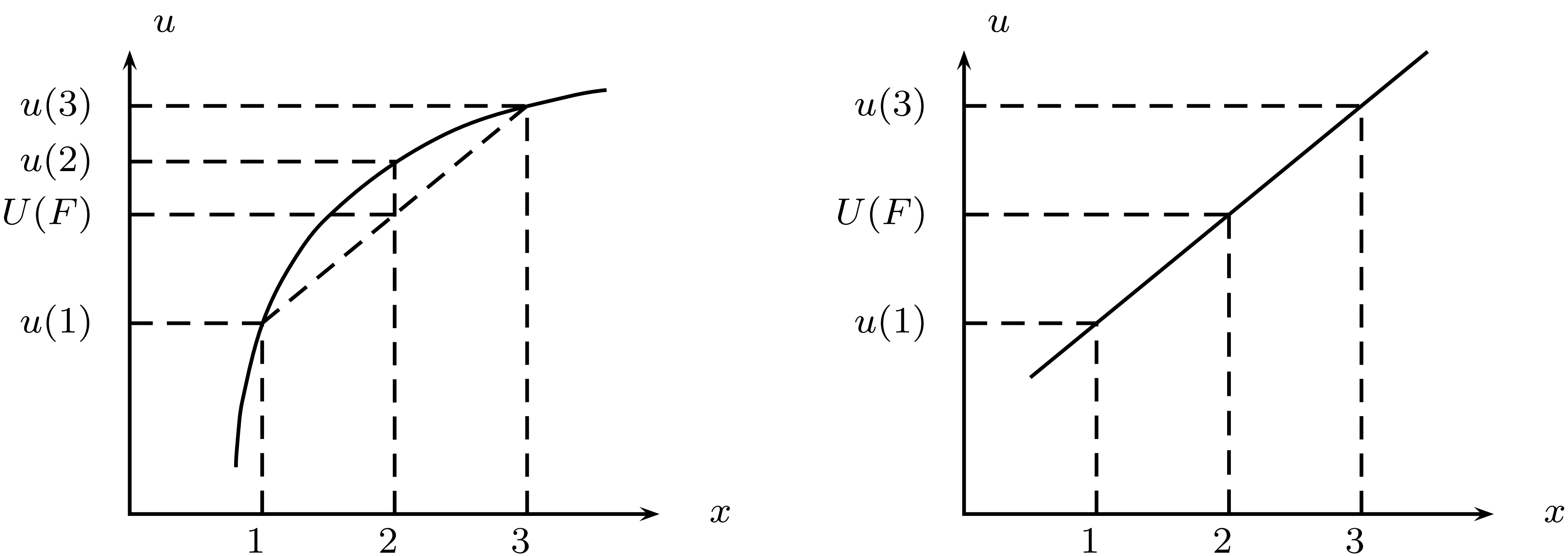

(a) Risk aversion

(b) Neutrality towards risk

Further readings:

- Gollier (2001) : “The Economics of Risk and Time”, Chapters 1, 2, 3 and 27
- Fishburn (1994) : “Utility and Subjective Probability”, in “Handbook of Game Theory” Vol. 2, Chap. 39
- Karni and Schmeidler (1991) : “Utility Theory with Uncertainty”, in “Handbook of Mathematical Economics” Vol. 4
- Kreps (1988) : “Notes on the Theory of Choice”
- Mas-Colell et al. (1995) : “Microeconomic Theory”, Sections 1.A, 1.B, 3.C, and Chapter 6
- Myerson (1991) : “Game Theory”, Chapter 1

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