Normal Form Games

(Static Games with Complete Information)

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Outline

(September 3, 2007)

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- Definitions and examples

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- Definitions and examples
- Nash Equilibrium
- Mixed Strategies
- Maxmin Strategies and Zero-Sum Games
- Iterated Elimination of Dominated Strategies

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$$s = (s_1, \ldots, s_n) \in S = S_1 \times \cdots \times S_n$$

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$$p(s_1 + s_2)\, s_i - \lambda_i\, s_i = s_i(a - b(s_1 + s_2) - \lambda_i)$$

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Normal form game: $N = \{1, 2\}$, $S_1 = S_2 = [0, 1]$, u_1 and u_2 above

A normal form game $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is finite if the sets of players and actions are finite

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2-player game:

$$
\begin{array}{c|c|c|c|}
 & \cdots & s_2 & \cdots \\
\hline
\vdots & \cdots & \cdots & \cdots \\
\hline
s_1 & \vdots & u_1(s_1, s_2); u_2(s_1, s_2) & \vdots \\
\hline
\vdots & \cdots & \cdots & \cdots \\
\hline
\end{array}
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2-player game:

	$\cdots$	s_2	$\cdots$
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s_1	$\vdots$	$u_1(s_1, s_2); u_2(s_1, s_2)$	$\vdots$
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3-player game with 2 actions per player:

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$\vdots$	$\cdots$	$\cdots$	$\cdots$

3-player game with 2 actions per player:

	s_2	s_2'
s_1	$u(s_1, s_2, s_3)$	$u(s_1, s_2', s_3)$
s_1'	$u(s_1', s_2, s_3)$	$u(s_1', s_2', s_3)$

s_3

	s_2	s_2'
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where $u(\cdot) = (u_1(\cdot),\, u_2(\cdot),\, u_3(\cdot))$

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	D	C
D	$(1,1)$	$(3,0)$
C	$(0,3)$	$(2,2)$

image

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➥ Dilemma between individual and collective rationality

✍ Consider the Cournot duopoly with $\lambda_1 = \lambda_2 = b = 1$ and $a = 4$ assuming that firm i can only choose between $s_i = 1$ (high production) and $s_i = 3/4$ (low production)

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Show that it is equivalent to the following prisoners dilemma

	High production	Low production
High production	$(10\,000,\ 10\,000)$	$(12\,500,\ \ 9\,375)$
Low production	$(\ 9\,375,\ 12\,500)$	$(11\,250,\ 11\,250)$

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✍ A variant of the prisoners dilemma with asymmetric players and where cooperation is always better for one player pdf

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$$\begin{array}{lcl} \forall\, s_{-i} \in S_{-i}, \quad u_i(s_i, s_{-i}) & \geq & u_i(s_i', s_{-i}) \\ \exists\, s_{-i} \in S_{-i}, \quad u_i(s_i, s_{-i}) & > & u_i(s_i', s_{-i}) \end{array}$$

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Example. In the following game H weakly dominates M, M weakly dominates B and H strictly dominates B. There is no dominance relation for player 2

	G	D
H	$(2,0)$	$(1,0)$
M	$(2,2)$	$(0,0)$
B	$(1,0)$	$(0,2)$

☞ Dominance is not sufficient to solve lots of games

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Coordination game.

	a	b
a	$(2, 2)$	$(0, 0)$
b	$(0, 0)$	$(1, 1)$

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	a	b
a	$(2, 2)$	$(0, 0)$
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Battle of sexes.

	a	b
a	$(3, 2)$	$(1, 1)$
b	$(0, 0)$	$(2, 3)$

Chicken game. image

	a	b
a	$(2, 2)$	$(1, 3)$
b	$(3, 1)$	$(0, 0)$

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	a	b
a	$(2,2)$	$(1,3)$
b	$(3,1)$	$(0,0)$

Stag hunt.

	a	b
a	$(3,3)$	$(0,2)$
b	$(2,0)$	$(1,1)$

Zero-Sum (Strictly Competitive) Games

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Matching pennies

	G	D
G	$(-1, 1)$	$(1, -1)$
D	$(1, -1)$	$(-1, 1)$

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	G	D
G	$(-1,1)$	$(1,-1)$
D	$(1,-1)$	$(-1,1)$

Paper, Rock, Scissors.

	P	R	S
P	$(0,0)$	$(1,-1)$	$(-1,1)$
R	$(-1,1)$	$(0,0)$	$(1,-1)$
S	$(1,-1)$	$(-1,1)$	$(0,0)$

Nash Equilibrium

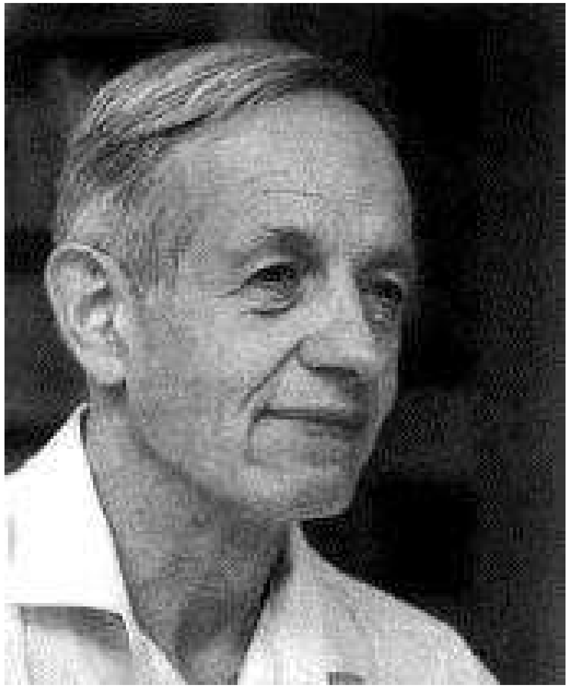

Figure 1: John F. Nash Jr (1928–)

Nash Equilibrium

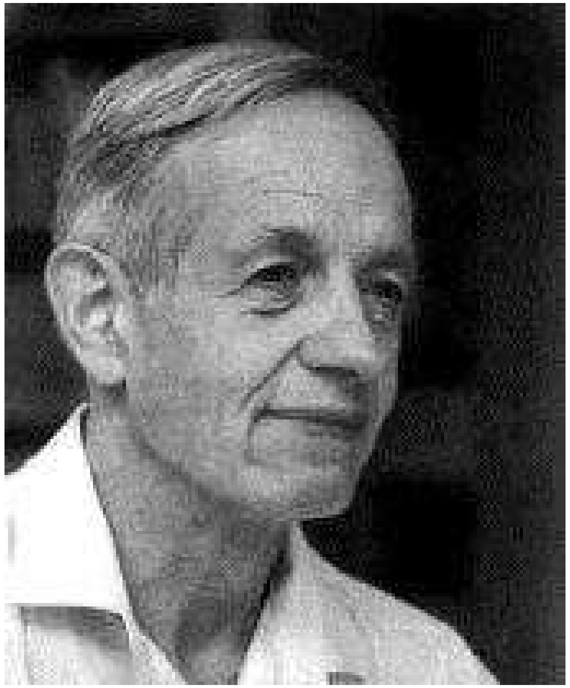

Figure 1: John F. Nash Jr (1928–)

Stability concept: situation in which no player has a unilateral incentive to deviate from his strategy

Definition. A Nash equilibrium (in pure strategies) of

$$\langle N,\, (S_i)_{i\in N},\, (u_i)_{i\in N}\rangle$$

is a profile of actions $s^* = (s_1^*, \ldots, s_n^*) \in S$ such that the action of each player is a best response to others actions, i.e.,

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If each player i strictly prefers action s_i^*, i.e.,

$$u_i(s_i^*, \, s_{-i}^*) > u_i(s_i, \, s_{-i}^*), \quad \forall \, s_i \neq s_i^*, \; \forall \, i \in N$$

then s^* is a strict Nash equilibrium

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Proof. ✍ (by definition) $\square$

✍ Nash equilibria and Pareto optimal solutions in the previous finite games?

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Example. Two players can share 2 euros. They simultaneously announce s_1, $s_2 \in [0, 2]$. If $s_1 + s_2 \leq 2$ then each player i gets the quantity s_i he asked for. Otherwise, if $s_1 + s_2 > 2$, they get nothing

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✍ Find a 3-action, 2-player game with exactly one Nash equilibrium in pure strategies, which is Pareto dominated and such that the strategies of both players are weakly dominated

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Concretely, how players coordinate their decisions on a specific equilibrium?

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➥ **Focal point** (Thomas C. Schelling, 1921–), Nobel prize in Economics in 2005 (with Robert J. Aumann) (image):

Equilibrium that players tend to play when they are not able to communicate because it seems natural, special or relevant to both of them

Application. International Negotiations / Public Good

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n countries negotiate their individual level of pollution $s_i \geq 0$. The payoff of country i is

$$u_i(s_1, \ldots, s_n) = v(s_i) - \sum_{j=1}^{n} s_j$$

where $v' > 0 > v''$ and $v'(0) > 1$

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Each player has a dominant action:

$$\frac{\partial u_i}{\partial s_i}(s) = 0 \quad \Leftrightarrow \quad v'(s_i) = 1$$

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Each player has a dominant action:

$$\frac{\partial u_i}{\partial s_i}(s) = 0 \quad \Leftrightarrow \quad v'(s_i) = 1$$

$\Rightarrow$ Unique and symmetric NE: each country chooses s_i^* such that $v'(s_i^*) = 1$. E.g., if $v(x) = \ln(x)$ then $s^* = (1, \ldots, 1)$

Action profile $\overline{s} = (\overline{s}_i)_i$ that maximizes social welfare

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$$\sum_{i=1}^{n} u_i(s_1, \ldots, s_n) = \sum_{i=1}^{n} v(s_i) - n \sum_{j=1}^{n} s_j$$

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is such that for every k,

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$\Rightarrow$ The NE is Pareto dominated

Action profile $\overline{s} = (\overline{s}_i)_i$ that maximizes social welfare

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$v'' < 0 \Rightarrow v' \searrow \Rightarrow s_i^* > \overline{s}_i$: at equilibrium, levels of pollution are too high

Tax rate θ :

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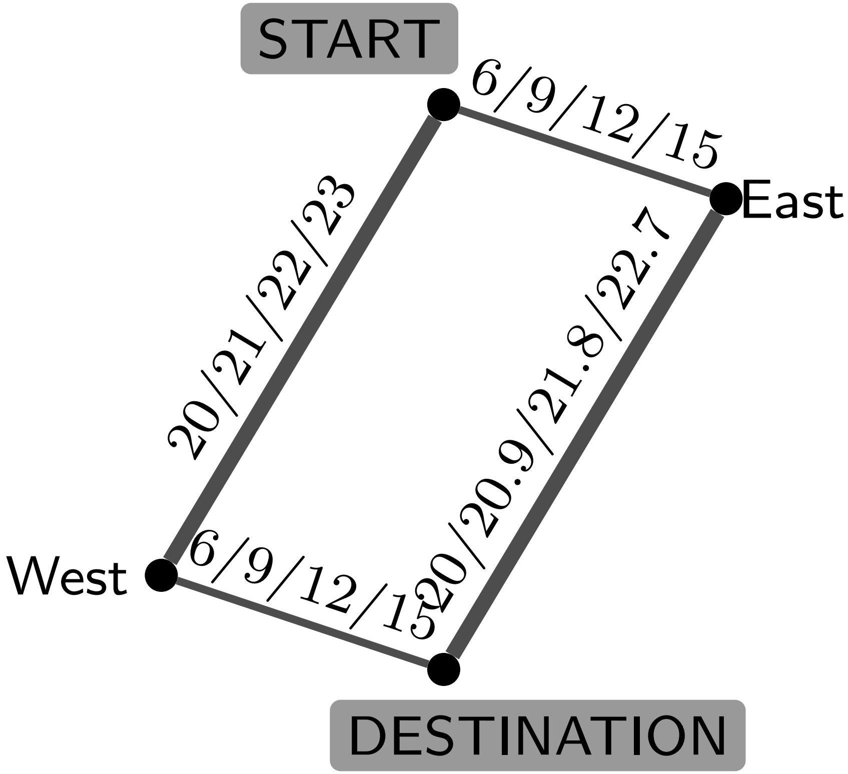

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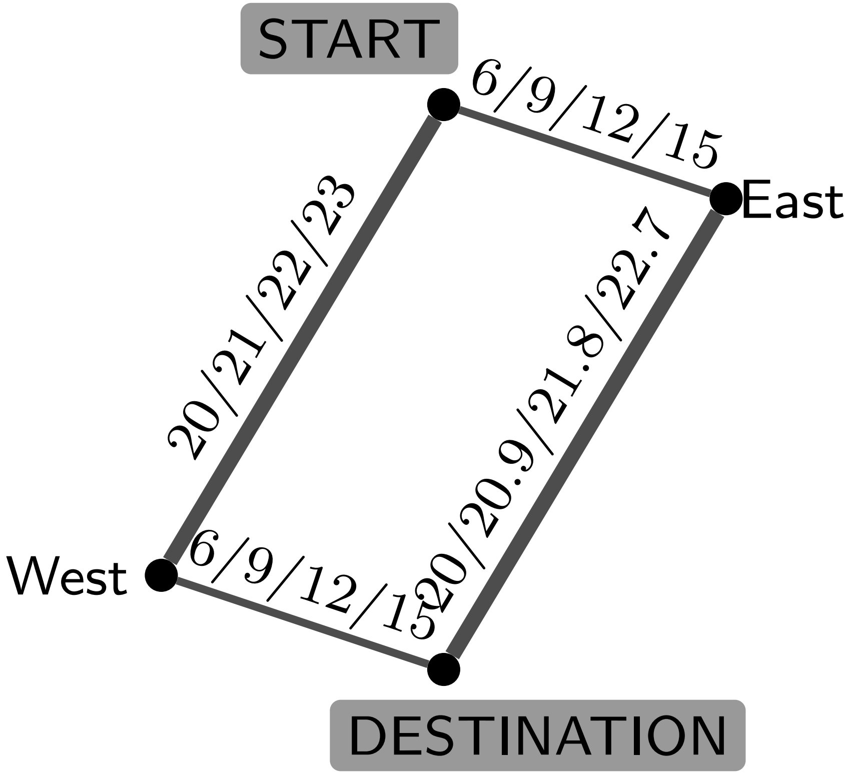

Nash equilibria: 2 drivers pass West and 2 drivers pass East, with 30 and 29.9 minutes travel time, respectively

New route (tunnel) from East to West (no change on the other routes)

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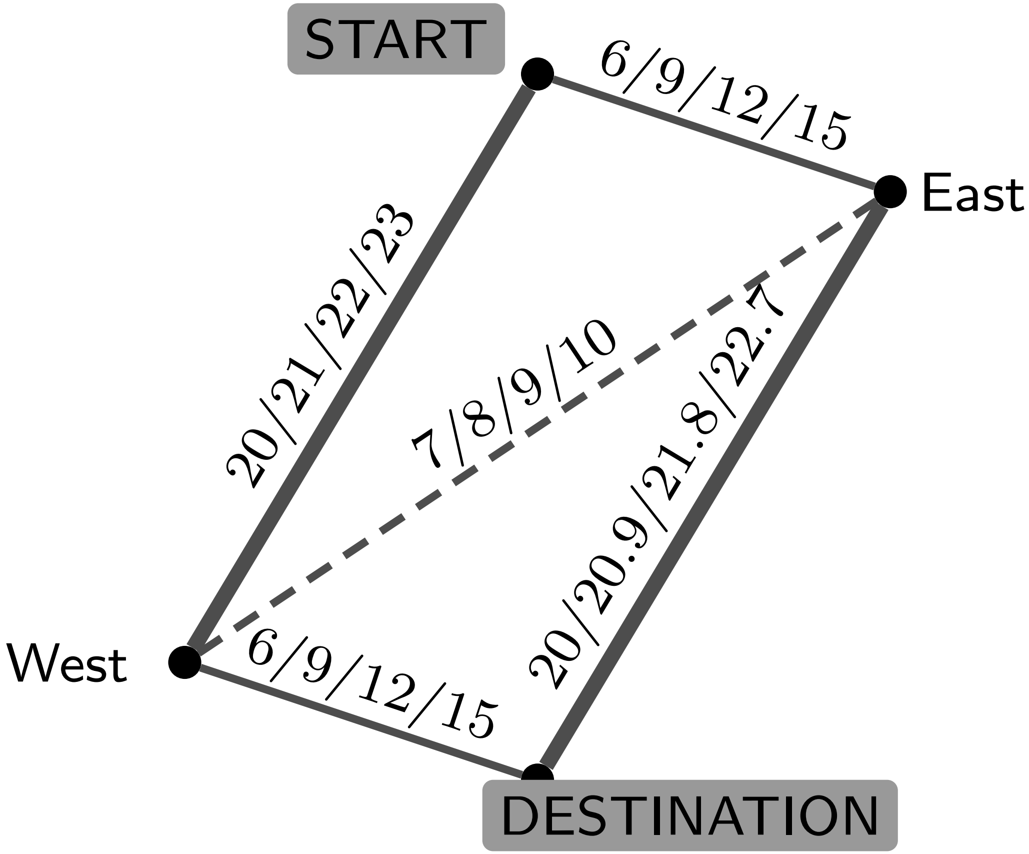

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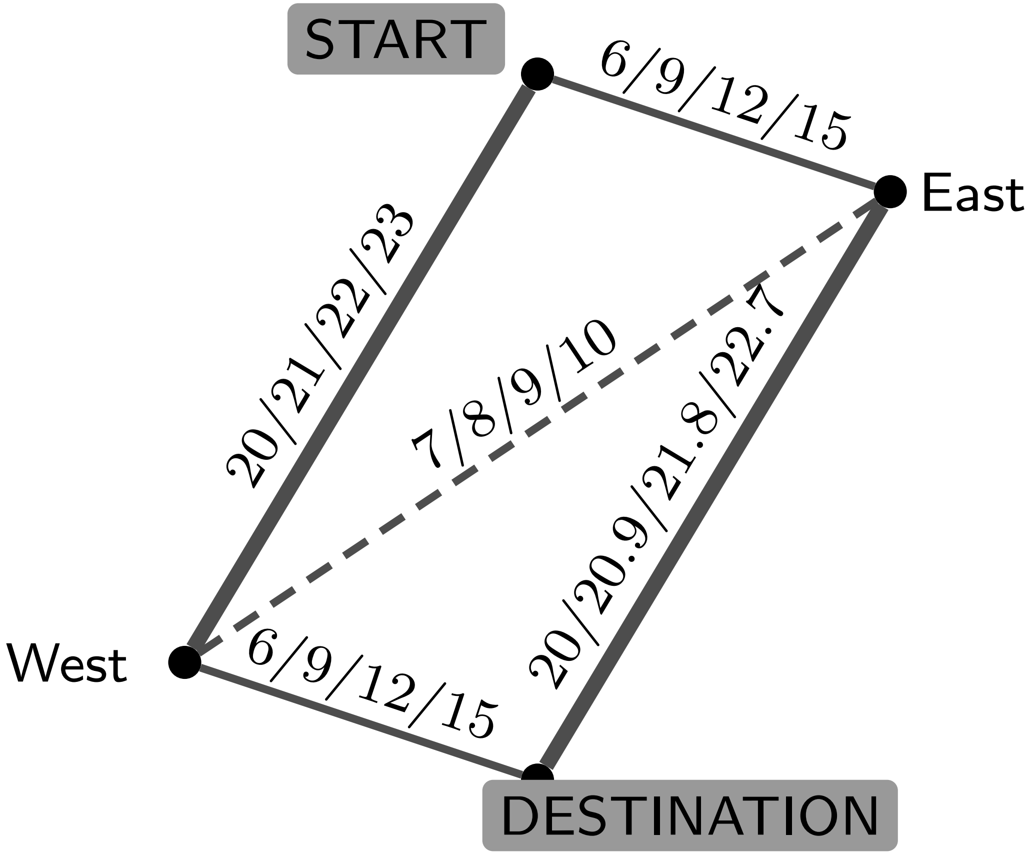

Nash equilibria: 2 drivers pass East and tunnel, 1 West, and 1 East

- Each driver's travel time is 32 minutes

➠ Each driver's travel time is 32 minutes

➠ The building of the tunnel, without modifying other routes' capacity, has increased the travel time of each driver!

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where $\mathrm{BR} : S \twoheadrightarrow S$ is defined by $\mathrm{BR}(s) = \mathrm{BR}_1(s_{-1}) \times \cdots \times \mathrm{BR}_n(s_{-n})$

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H	$1\ ,\ 2^*$	$2^*,\ 1$	$1^*,\ 0$
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Two $*$ $\leftrightarrow$ each player plays a best response to his opponent's strategy

$\leftrightarrow$ Nash equilibrium (here, (M, G) and (B, D))

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Proof. Apply Kakutani's (1941) fixed point theorem to the correspondence $\mathrm{BR} : S \twoheadrightarrow S$ $\square$

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$$u_i(s_1, s_2) = s_i(-\theta_i - s_1 - s_2)$$

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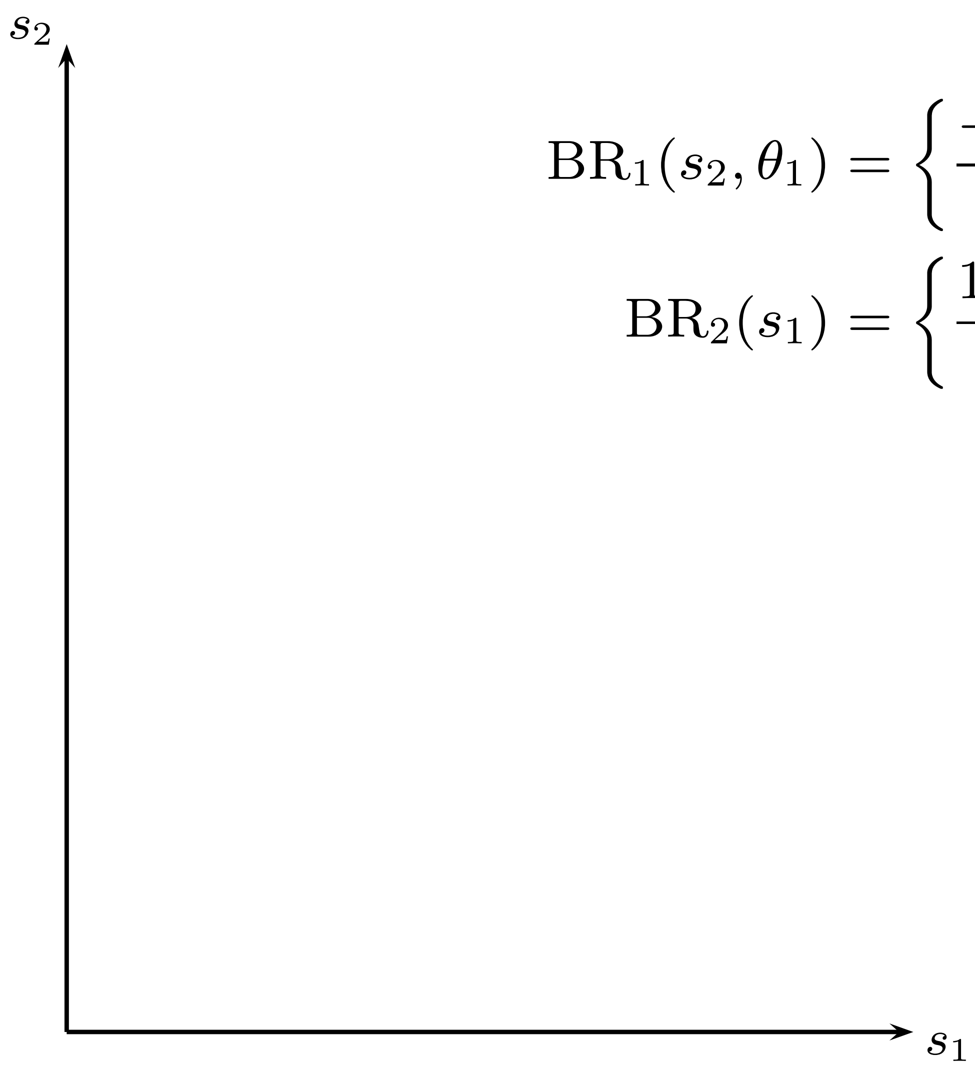

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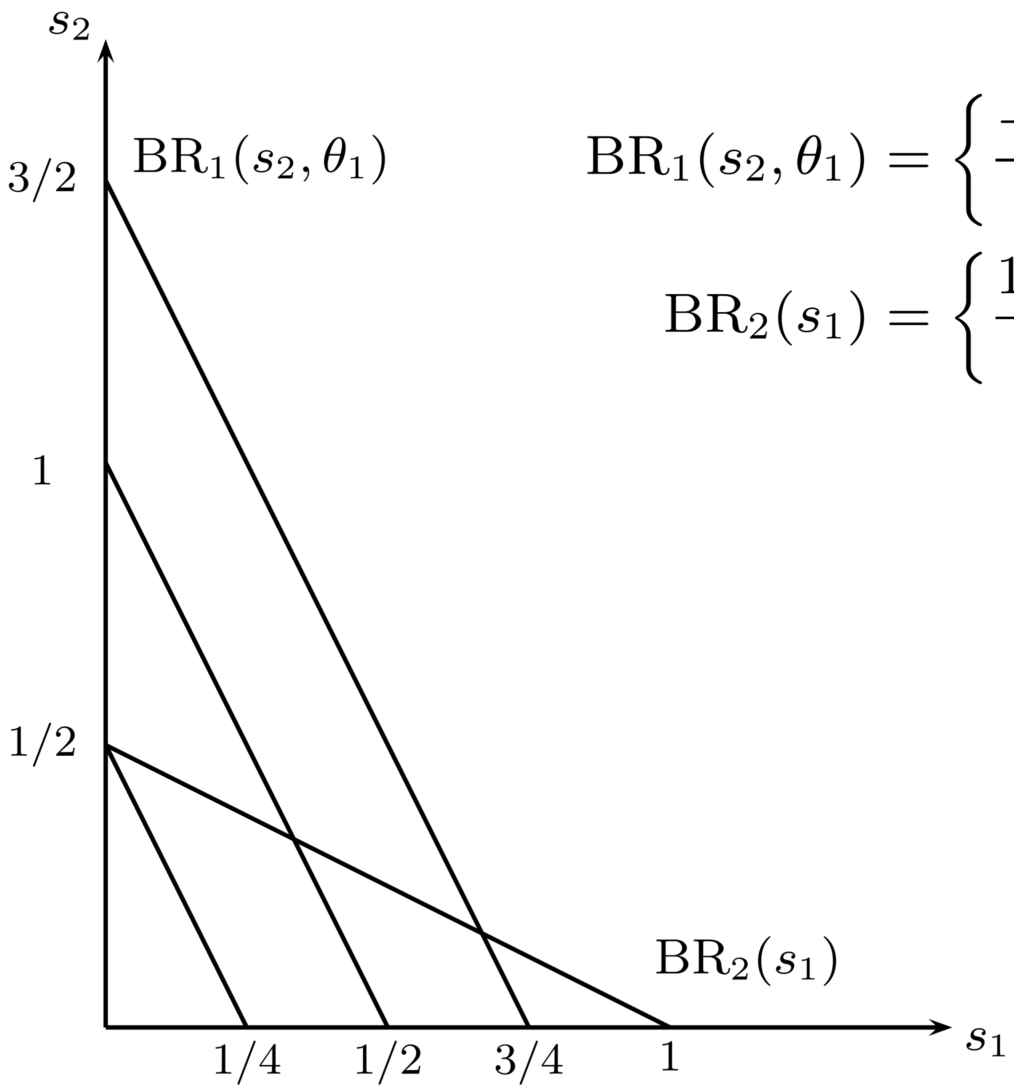

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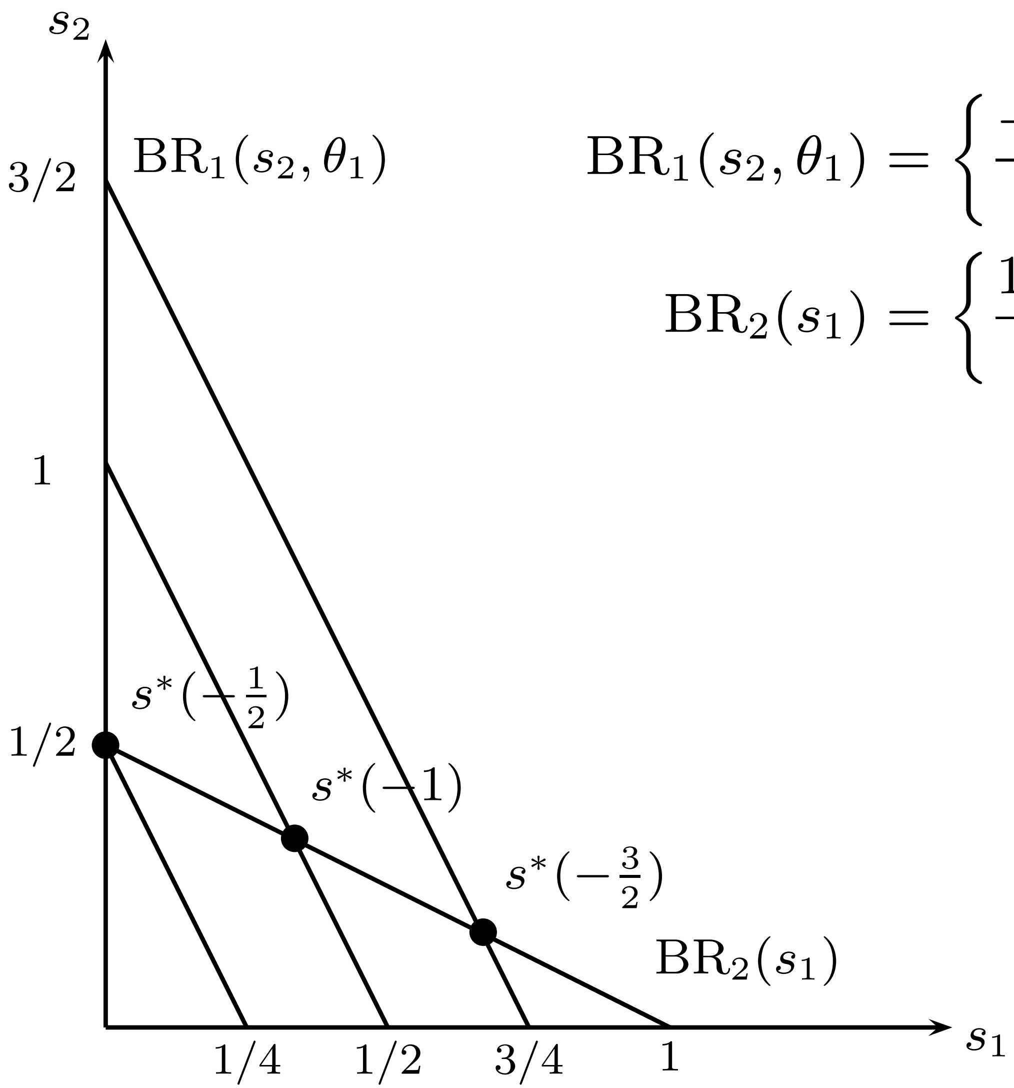

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Proof. $\mathrm{BR}_1(a) = \mathrm{BR}_2(a) = f(a)$ for all $a \in A$. Then, apply Kakutani's fixed point theorem to $f : A \twoheadrightarrow A$. $\Rightarrow$ there exists a^* such that $a^* \in f(a^*)$. Hence, (a^*, a^*) is a Nash equilibrium because $a^* \in \mathrm{BR}_i(a^*)$, $i = 1$, 2 $\square$

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Remark. Some equilibria of a symmetric game may be asymmetric (see, e.g., the chicken game)

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However, there is a unique Nash equilibrium: $p_1^* = p_2^* = c$ (perfectly competitive price, zero profit)