Mixed Extension of a Game

(September 3, 2007)

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Pure strategy (action): often insufficient to appropriately describe players' behavior

How to formalize the idea that a player chooses **more often** Rock than Scissors?

➜ Best response of his opponent: play more often Paper

➜ The first player must choose more often Scissors, . . .

➥ Can behavior stabilize?

☞ We must define **mixed strategies**

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- ➢ ruse
- ➢ secret
- ➢ bluffing

Ex : penalty kick, poker, rock-paper-scissors, tax inspections . . . image

Definition. A **mixed strategy** for player i is a probability distribution over the set S_i of pure strategies of player i

$$\Sigma_i = \Delta(S_i) \equiv \{p \in \mathbb{R}^{|S_i|} : p_k \geq 0,\ \sum_k p_k = 1\}$$

is the set of mixed strategies for player i. Let $\sigma_i = (\sigma_i(s_i))_{s_i \in S_i}$ be an element of Σ_i

A mixed strategy σ_i is *totally mixed* if $\sigma_i(s_i) > 0$ for every $s_i \in S_i$

3/ VNM preferences $\Rightarrow$ $\sigma = (\sigma_i)_{i \in N}$ is evaluated by player i with the expected utility function $U_i : \Sigma \to \mathbb{R}$

$$U_i(\sigma) = \sum_{s \in S} \sigma(s)\, u_i(s)$$

where $\sigma(s)$ is the probability that the profile of actions s is played given σ

Independent strategies $\Rightarrow$ $\sigma(s) = \prod_{i \in N} \sigma_i(s_i)$

Example. 2-player, 2-action games: $S_1 = S_2 = \{a, b\}$, $p = \sigma_1(a)$, $q = \sigma_2(a)$

	a	b	
a	$p\,q$	$p\,(1-q)$	p
b	$(1-p)\,q$	$(1-p)\,(1-q)$	$1-p$
	q	$1-q$	

$$U_i(\sigma) = \ p\,q\,u_i(a,\,a) + p\,(1-q)\,u_i(a,\,b) + (1-p)\,q\,u_i(b,\,a) + (1-p)\,(1-q)\,u_i(b,\,b)$$

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The normal form game $\langle N, (\Sigma_i)_i, (U_i)_i \rangle$ is called the **mixed extension** of the normal form game $\langle N, (S_i)_i, (u_i)_i \rangle$

In the following, "$u_i = U_i$"

➥ $u_i(s_i, \sigma_{-i}) =$ expected utility of player i when he plays the pure strategy s_i and the other players choose the mixed strategy profile σ_{-i}

Definition. A **mixed strategy Nash equilibrium** of the normal form game

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$$\langle N, (S_i)_i, (u_i)_i \rangle$$

is a pure strategy equilibrium of its mixed extension

that is, a profile of strategies $\sigma^* \in \Sigma$ such that

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \quad \forall\, \sigma_i \in \Sigma_i, \ \forall\, i \in N$$

Illustration : Individual Decision

Choose between $a =$ "jogging", $b =$ "homework"
$c =$ "cinema", $d =$ "siesta"

$\sigma_i = (p_a, p_b, p_c, p_d)$: Strategy of the decisionmaker

$\Rightarrow$ Lottery

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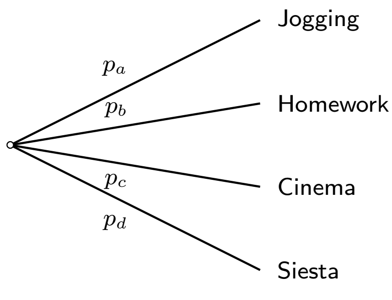

The decisionmaker chooses $\sigma_i = (\frac{5}{6}, \frac{1}{6}, 0, 0)$

$$\Rightarrow \operatorname{supp}[\sigma_i] = \{a, b\}$$

For every (p_a, p_b, p_c, p_d),

$$\frac{5}{6}u_i(a) + \frac{1}{6}u_i(b) \geq p_a\, u_i(a) + p_b\, u_i(b) + p_c\, u_i(c) + p_d\, u_i(d)$$

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$$\Leftrightarrow \quad u_i(a) = u_i(b) \geq \begin{cases} u_i(c) \\ u_i(d) \end{cases}$$

$\Rightarrow$ if the decision maker "throws a dice" to choose between a and b, then he should prefer a and b to all other actions and should be indifferent between a and b

Proposition. $\sigma^* \in \Sigma$ *is a* Nash equilibrium in mixed strategies *of a finite normal form game if and only if for all* $i \in N$

$$u_i(s_i,\, \sigma^*_{-i}) = u_i(s'_i,\, \sigma^*_{-i}), \quad \forall\, s_i,\, s'_i \in \mathrm{supp}[\sigma^*_i]$$
$$\textit{and} \quad u_i(s_i,\, \sigma^*_{-i}) \geq u_i(s''_i,\, \sigma^*_{-i}), \quad \forall\, s_i \in \mathrm{supp}[\sigma^*_i],\; s''_i \in S_i$$

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Proof. Directly from the multi-linearity of the expected utility function:

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i)\, u_i(s_i, \sigma_{-i})$$

In particular, $\sigma_i \in \mathrm{BR}_i(\sigma_{-i})$ if and only if $s_i \in \mathrm{BR}_i(\sigma_{-i})$ for all $s_i \in \mathrm{supp}[\sigma_i]$. That is, $\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$ $\square$

✍ Verify and explain why $\sigma_1 = (3/4, 0, 1/4)$ and $\sigma_2 = (0, 1/3, 2/3)$ is a Nash equilibrium (a point $\cdot$ can be any payoff)

	G (0)	C (1/3)	D (2/3)
H (3/4)	$(\cdot, 2)$	$(3, 3)$	$(1, 1)$
M (0)	$(\cdot, \cdot)$	$(0, \cdot)$	$(2, \cdot)$
B (1/4)	$(\cdot, 4)$	$(5, 1)$	$(0, 7)$

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Proposition. (Nash Theorem) *Every finite normal form game has at least one Nash equilibrium in mixed strategies*

Proof. It is a corollary of the existence theorem of Nash equilibrium in pure strategies since the mixed extension of a finite game satisfies the required assumptions:

➢ for every i, the set of strategies $\Sigma_i \subseteq \mathbb{R}^{|S_i|}$ is non-empty, compact and convex (it is the simplex of dimension $|S_i|$)

10/ ➢ the function $u_i : \Sigma \to \mathbb{R}$ is multi-linear, so $u_i(\sigma)$ is continuous in σ and quasi-concave in σ_i. □

Proposition. *Every symmetric and finite game has as symmetric mixed strategy Nash equilibrium*

Proof. Directly from the proposition on the existence of a symmetric pure strategy equilibrium □

Examples

Prisoners dilemma.

	D	C
D	$(1,1)$	$(3,0)$
C	$(0,3)$	$(2,2)$

11/ (D,D) is the unique Nash equilibrium because D strictly dominates C for both players

Coordination game.

	a	b	
a	$(2,2)$	$(0,0)$	p
b	$(0,0)$	$(1,1)$	$1-p$
	q	$1-q$	

$p=\sigma_1(a)$: probability that player 1 chooses action a

$q=\sigma_2(a)$: probability that player 2 chooses action a

If $p\in\{0,1\}$ we get the two pure strategy NE (a,a) and (b,b)

12/ If $0<p<1$ then player 1 should be indifferent

$$a \xrightarrow{1} u_1(a,\sigma_2)=2q+0(1-q)=2q$$

$$b \xrightarrow{1} u_1(b,\sigma_2)=0q+1(1-q)=1-q$$

so $a\sim_1 b \Leftrightarrow 2q=1-q \Leftrightarrow q=\frac{1}{3}$

Player 2 plays $\sigma_2(a)=q=1/3$ if he is also indifferent. By symmetry, $p=\frac{1}{3}$

$\Rightarrow$ Non degenerate mixed strategy NE:

$$\sigma = \left(\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \right)$$

$\Rightarrow$ 3 NE, including 2 in pure strategies

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Best responses correspondences:

$$\mathrm{BR}_i(\sigma_j) = \begin{cases} \{0\} & \text{if } \sigma_j(a) < \frac{1}{3} \\ [0,1] & \text{if } \sigma_j(a) = \frac{1}{3} \\ \{1\} & \text{if } \sigma_j(a) > \frac{1}{3} \end{cases}, \quad i = 1, 2,\ j \neq i$$

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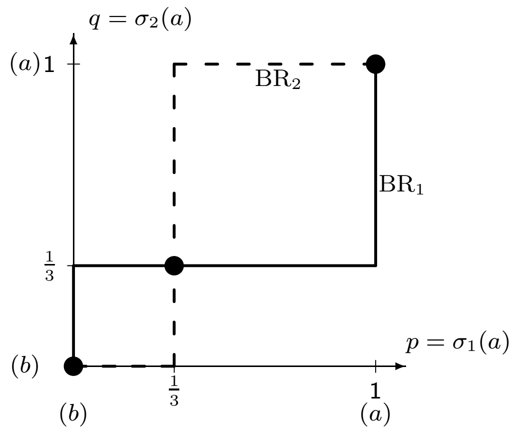

Battle of sexes.

	a	b	
a	$(3,2)$	$(1,1)$	p
b	$(0,0)$	$(2,3)$	$1-p$
	q	$1-q$	

$$a \xrightarrow{1} 3q + (1-q) = 1 + 2q$$
$$b \xrightarrow{1} 2(1-q) = 2 - 2q$$

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thus $a \sim_1 b \Leftrightarrow 1 + 2q = 2 - 2q \Leftrightarrow q = \frac{1}{4}$

$$a \xrightarrow{2} 2p$$
$$b \xrightarrow{2} p + 3(1-p) = 3 - 2p$$

thus $a \sim_2 b \Leftrightarrow p = \frac{3}{4}$

	a	b	
a	$(3,2)$	$(1,1)$	p
b	$(0,0)$	$(2,3)$	$1-p$
	q	$1-q$	

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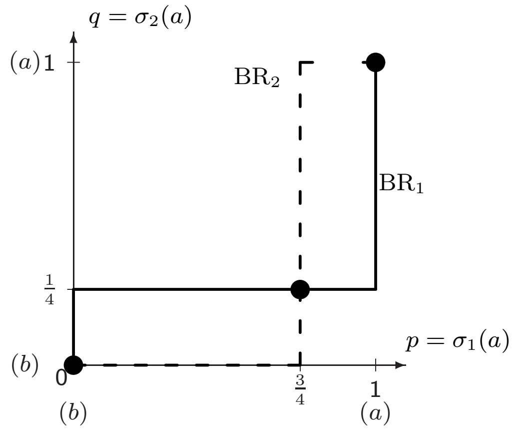

Chicken game.

	a	b	
a	$(2,2)$	$(1,3)$	p
b	$(3,1)$	$(0,0)$	$1-p$
	q	$1-q$	

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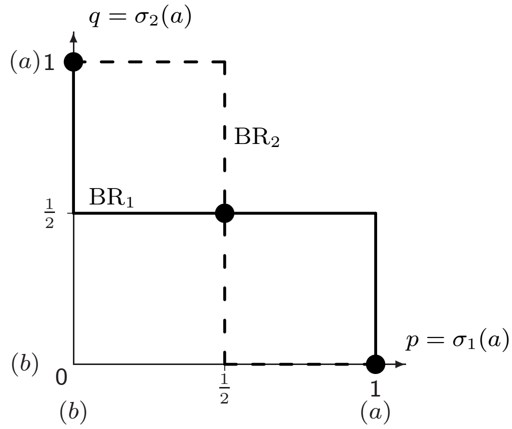

Stag hunt.

	a	b	
a	$(3,3)$	$(0,2)$	p
b	$(2,0)$	$(1,1)$	$1-p$
	q	$1-q$	

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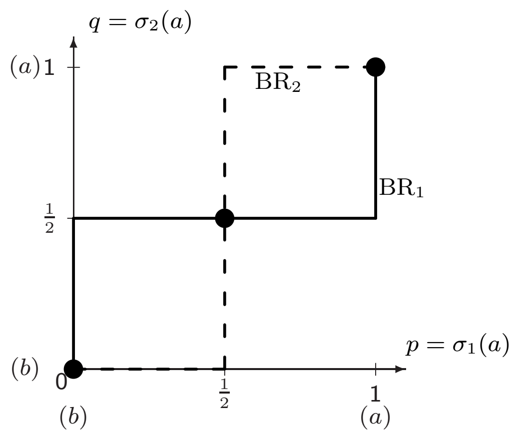

Matching pennies.

	G	D	
G	$(-1,1)$	$(1,-1)$	p
D	$(1,-1)$	$(-1,1)$	$1-p$
	q	$1-q$	

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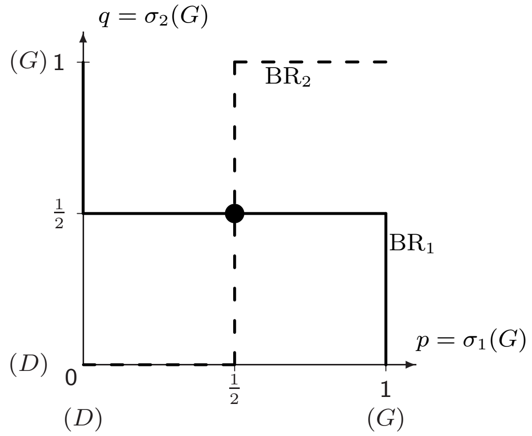

Paper, Rock, Scissors.

	F	P	C	
F	$(0,0)$	$(1,-1)$	$(-1,1)$	a
P	$(-1,1)$	$(0,0)$	$(1,-1)$	b
C	$(1,-1)$	$(-1,1)$	$(0,0)$	c
	p	q	r	

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$$a = b = c = 1/3 \text{ and } p = q = r = 1/3$$

Reporting a Crime

New York Times, March 1964 :

"*37 Who Saw Murder Didn't Call the Police*"

We sometimes observe that when the number of witnesses increases, there is a decline not only in the probability that any given individual intervenes, but also in the probability that at least one of them intervenes!

21/ Three main explanations are given:

- ➢ diffusion of responsibility
- ➢ audience inhibition
- ➢ social influence

☛ All these factors raise the expected cost and/or reduce the expected benefit of a person's intervening

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A game theoretical explanation.

• n players (witnesses)

• Two actions : call the Police (action P) or do Nothing (action N)

• Preferences : value v if police is called, cost c if the individual calls the police himself, where

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$$v > c > 0$$

➥ n NE in pure strategies (exactly one person calls the police)

But coordination to such an asymmetric equilibrium is difficult without communication or convention

Symmetric equilibrium here: in (non-degenerated) mixed strategies

$\sigma_i(P) = q \in (0,1)$: probability that player i calls the police, $i = 1, \ldots, n$

☞ Each player should be indifferent between P and N

$$P \xrightarrow{i} v - c$$

$$N \xrightarrow{i} 0 \ \mathrm{Pr}(\text{nobody calls}) + v \ \mathrm{Pr}(\text{at least one calls})$$

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so $P \sim_i N \ \Leftrightarrow \ v - c = v[1 - (1-q)^{n-1}]$

$$\Rightarrow \quad q = 1 - (c/v)^{1/n-1}$$

✔ $\mathrm{Pr}(\text{one given person calls}) = 1 - (c/v)^{1/n-1}$ decreases with n

✔ $\mathrm{Pr}(\text{at least one person calls}) = 1 - (1-q)^n = 1 - (c/v)^{n/n-1}$ decreases with n !

Reporting a crime when the witnesses are heterogeneous.

✍ Consider a variant of the model in which n_1 witnesses incur the cost c_1 to report the crime, and n_2 witnesses incur the cost c_2, where $0 < c_1 < v$, $0 < c_2 < v$, and $n_1 + n_2 = n$. Show that if c_1 and c_2 are sufficiently close, then the game has a mixed strategy Nash equilibrium in which every witness's strategy assigns positive probabilities to both reporting (calling the police) and not reporting.

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Finding all Nash Equilibria

	A	B	C
a	$3, 2$	$1, 1$	$0, 1.5$
b	$0, 0$	$2, 3$	$1, 2$

Figure 1: A variant of the battle of sexes

Two pure strategy Nash equilibria: (a, A) and (b, B)

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How to find all (mixed strategy) equilibria?

☞ Consider every possible equilibrium support for player 1, and in each case consider every possible support for player 2

We find $(\sigma_1, \sigma_2) = ((4/5, 1/5),\ (1/4, 0, 3/4))$

Prudent / Maxmin Strategy

2-player finite games $\langle \{1,2\}, (S_1, S_2), (u_1, u_2) \rangle$ image

Payoff guaranteed by strategy $s_1 \in S_1$ of player 1 = worst payoff that player 1 can get by playing action s_1 :

$$\eta_1(s_1) = \min_{\sigma_2 \in \Sigma_2} u_1(s_1, \sigma_2) = \min_{s_2 \in S_2} u_1(s_1, s_2)$$

27/ Example : Chicken game

	a	b
a	$(2,2)$	$(\mathbf{1},3)$
b	$(3,\mathbf{1})$	$(\mathbf{0},\mathbf{0})$

$$\eta_1(a) = \eta_2(a) = 1$$
$$\eta_1(b) = \eta_2(b) = 0$$

A maxmin action is an action that maximizes the payoff that the player can guarantee ⇒ “maxminimizing” action

Definition. An action $s_1^* \in S_1$ is a **maxmin** or **maxminimizing** action for player 1 if

$$s_1^* \in \arg\max_{s_1 \in S_1} \eta_1(s_1) = \arg\max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2)$$

$\max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2)$: *maxminimized payoff in pure strategies* for player 1, i.e., the maximum payoff player 1 can guarantee in pure strategies

Example : in the chicken game

	a	b
a	$(2,2)$	$(\mathbf{1},3)$
b	$(3,\mathbf{1})$	$(\mathbf{0},\mathbf{0})$

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- maxmin strategy of each player: a
- maxminimized payoff: 1

☞ A maxmin strategy profile is not necessarily a Nash equilibrium

Definition. A mixed strategy $\sigma_1^* \in \Sigma_1$ is a **maxmin (mixed) strategy** for player 1 if

$$\sigma_1^* \in \arg\max_{\sigma_1 \in \Delta(S_1)} \eta_1(\sigma_1) = \arg\max_{\sigma_1 \in \Delta(S_1)} \min_{s_2 \in S_2} u_1(\sigma_1, s_2)$$

$\max_{\sigma_1 \in \Sigma_1} \min_{s_2 \in S_2} u_1(\sigma_1, s_2)$: *maxminimized payoff in mixed strategies* for player 1, i.e., the maximum payoff player 1 can guarantee in mixed strategies

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In the chicken game maxmin mixed strategy = maxmin pure strategy

	a	b
a	$(2,2)$	$(1,3)$
b	$(3,1)$	$(0,0)$

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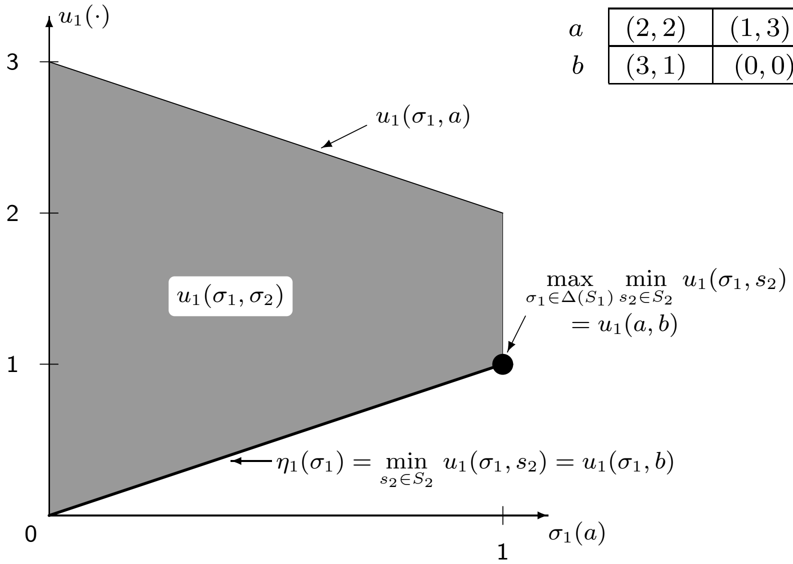

But a maxmin strategy is not necessarily a pure strategy. Battle of sexes

	a	b
a	$(3,2)$	$(1,1)$
b	$(0,0)$	$(2,3)$

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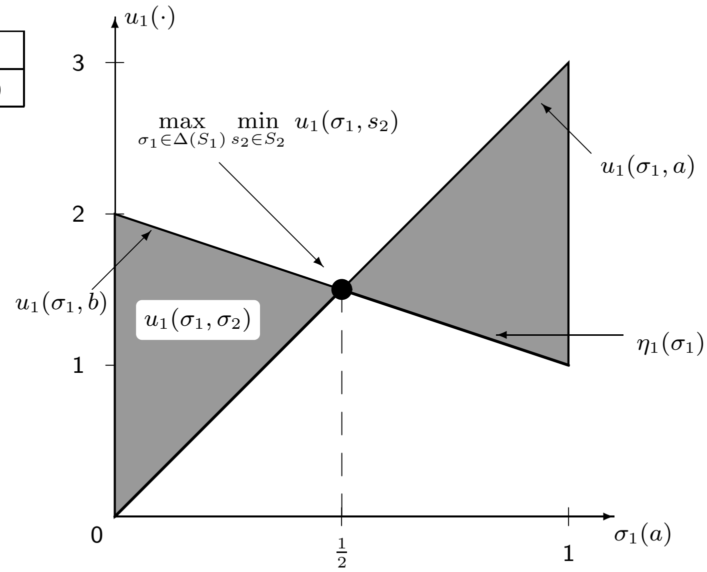

$$\max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2) = 1 < \max_{\sigma_1 \in \Sigma_1} \min_{s_2 \in S_2} u_1(\sigma_1, s_2) = 1.5$$

Zero-Sum Games

Definition. A 2-player game $\langle \{1,2\}, (S_1, S_2), (u_1, u_2) \rangle$ is a **zero-sum game** or **strictly competitive game** if players' preference are diametrically opposed: $u_1 = u$ and $u_2 = -u$

Remark. In such games every outcome is clearly Pareto optimal

Examples : matching pennies, rock-paper-scissors, chess

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✍ Find other examples of 0-sum games

Theorem. *In a finite zero-sum game, (σ_1^*, σ_2^*) is a Nash equilibrium if and only if (σ_1^*, σ_2^*) is a maxmin strategy profile. In addition, we have*

$$u_1(\sigma_1^*, \sigma_2^*) = \max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2) = \min_{\sigma_2 \in \Sigma_2} \max_{\sigma_1 \in \Sigma_1} u_1(\sigma_1, \sigma_2). \tag{1}$$

Hence, every Nash equilibrium gives the same payoff to player 1 (his maxminimized payoff, also called the **value** *of the game), and to player 2*

Remarks.

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- ➢ Equality (1) $\sim$ Maxmin theorem (von Neumann, 1928)
- ➢ Equality (1) is not true in pure strategies. Ex : matching pennies $\max_{s_1 \in S_1} \min_{s_2 \in S_2} u_1(s_1, s_2) = -1 < \min_{s_2 \in S_2} \max_{s_1 \in S_1} u_1(s_1, s_2) = 1$
- ➢ The equilibrium payoff can be guaranteed independently of the opponent's strategy $\Rightarrow$ *optimal strategy*
- ➢ Equilibrium strategies are *interchangeable*

Example. In matching pennies the maxmin strategy of player 1 is indeed equivalent to his equilibrium strategy $\sigma_1^*(G) = 1/2$

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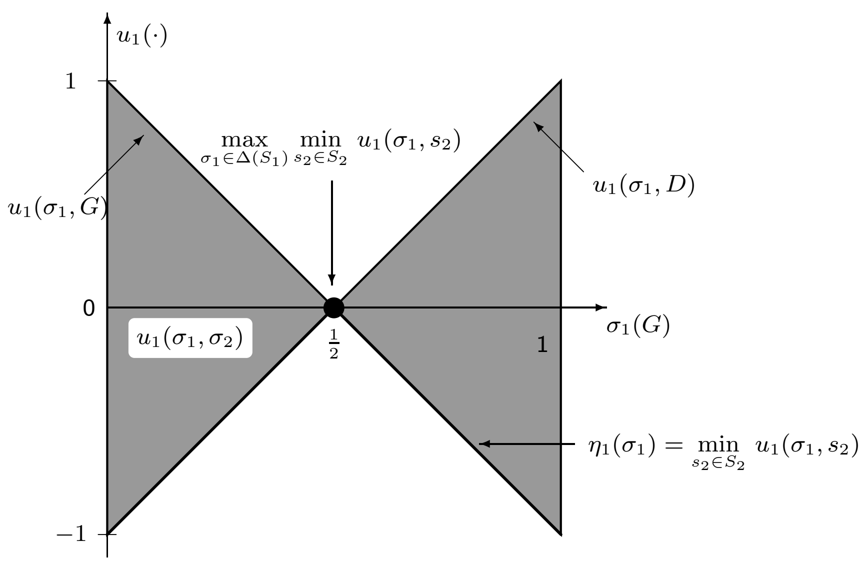

Proposition.

Let G be a finite zero-sum game and G' the game obtained from G by deleting an action of player i

Then the payoff of player i in G' is not larger than his equilibrium payoff in G

This is not necessarily true in non zero-sum games, even if the equilibrium is unique

Proof. Directly from the fact that in zero-sum games

35/ $u_1(\sigma_1^*, \sigma_2^*) = \max_{\sigma_1 \in \Sigma_1} \min_{\sigma_2 \in \Sigma_2} u_1(\sigma_1, \sigma_2)$ from the theorem, and the fact that $Y \subseteq X$ implies $\max_{x \in X} f(x) \geq \max_{x \in Y} f(x)$

In non zero-sum games, consider

	a	b
a	$(10, 0)$	$(1, 1)$
b	$(5, 5)$	$(0, 0)$

The unique Nash equilibrium is (a, b). If we delete a for player 1 then the unique equilibrium becomes (b, a)

36/ ✍ Explain why in a finite symmetric zero-sum game players' payoff is zero in every Nash equilibrium

Iterated Elimination of Dominated Strategies

- **Nash Equilibrium**: rational expectation, perfect coordination
- **Maxmin strategy**: pessimist, naive expectation (except in zero-sum games)
- **Iterated elimination of dominated strategies**: no ad hoc assumptions on

37/ players' expectations but
  - each player is rational
  - each player thinks that others are rational
  - each player thinks that others think that others are rational ...

  Common knowledge of rationality

Definition. A pure strategy $s_i \in S_i$ is **strictly dominated** if there is a mixed strategy $\sigma_i \in \Sigma_i$ such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$

38/ A pure strategy $s_i \in S_i$ is **weakly dominated** if there is a mixed strategy $\sigma_i \in \Sigma_i$ such that $u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$, with a strict inequality for at least one s_{-i}

A strategy can be strictly dominated by a mixed strategy without being strictly dominated by any pure strategy

Example.

	G	D
H	$(3,0)$	$(0,1)$
M	$(0,0)$	$(3,1)$
B	$(1,1)$	$(1,0)$

39/ B is strictly dominated by $(1/2)H + (1/2)M$ but not by H or M

Remark. If s_i is strictly (weakly) dominated then every mixed strategy putting strictly positive probability on s_i is strictly (weakly) dominated

A player does not play a strictly dominated strategy if and only if he maximizes his expected payoff given some beliefs about others' (possibly correlated) strategies

Proposition. *A strategy s_i of player i is strictly dominated if and only if s_i is never a best response, i.e., $s_i \notin \mathrm{BR}_i(\mu_{-i})$ for any belief $\mu_{-i} \in \Delta(S_{-i})$ of player i about others' behavior*

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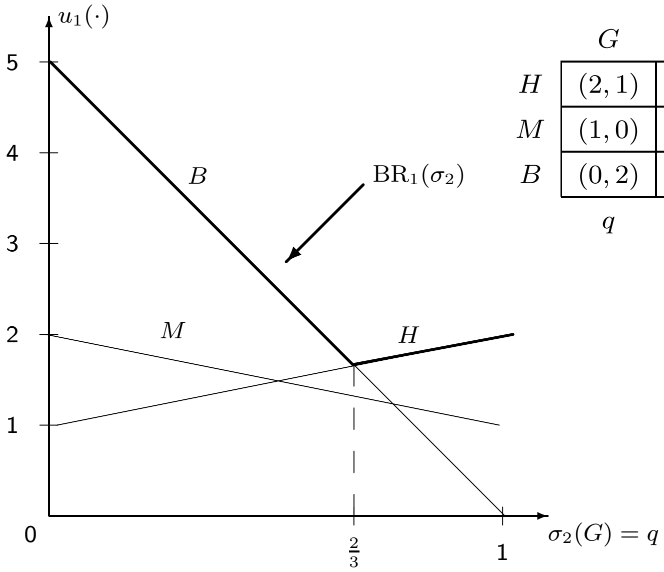
Example.

	G	D
H	$(2,1)$	$(1,2)$
M	$(1,0)$	$(2,5)$
B	$(0,2)$	$(5,1)$
	q	$1-q$



	G	D
H	$(2,1)$	$(1,2)$
M	$(1,0)$	$(2,5)$
B	$(0,2)$	$(5,1)$
	q	$1-q$

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M is never a best response, so it is strictly dominated (✍ by which strategy?)

Definition. A set of strategy profiles $S^* \subseteq S$ **survives iterated elimination of strictly dominated strategies** if there is a sequence $(S^k)_{k=1}^K$, where $S^k = (S_i^k)_{i \in N}$ for every k, such that

- $S^0 = S$ and $S^K = S^*$
- $S^{k+1} \subseteq S^k$ for every k
- if $s_i \in S_i^k$ but $s_i \notin S_i^{k+1}$ then s_i is strictly dominated in the reduced game $G^k = \langle N, (S_i^k)_i, (u_i)_i \rangle$
- 42/ no action is strictly dominated in the reduced game $G^K = \langle N, (S_i^K)_i, (u_i)_i \rangle$

Example.

	G	D
H	$(3,0)$	$(0,1)$
M	$(0,0)$	$(3,1)$
B	$(1,1)$	$(1,0)$

➠

	G	D
H	$(3,0)$	$(0,1)$
M	$(0,0)$	$(3,1)$

➠

	D
H	$(0,1)$
M	$(3,1)$

➠

	D
M	$(3,1)$

Proposition. *The set S^* that survives iterated elimination of strictly dominated strategies is uniquely defined*

$\Rightarrow$ the ordering of elimination does not influence the final outcome

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The previous result is not true for iterated elimination of *weakly* dominated strategies

Example.

	G	D
H	$(1,1)$	$(0,0)$
M	$(1,1)$	$(2,1)$
B	$(0,0)$	$(2,1)$

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Proposition. *Any action played with positive probability at a Nash equilibrium survives iterated elimination of strictly dominated strategies. This is not necessarily true for iterative elimination of weakly dominated strategies. However, after iteratively eliminating weakly dominated strategies, there is always at least one Nash equilibrium of the original game that survived*

Example. Cournot duopoly with $\theta_1 = \theta_2 = -1$:

$$u_i(s_1, s_2) = s_i(1 - s_1 - s_2) \qquad \mathrm{BR}_i(s_j) = \left\{\frac{1 - s_j}{2}\right\}$$

Only the NE survives iterated elimination of strictly dominated strategies

$$S_i^1 = \mathrm{BR}_i([0,1]) = [\mathrm{BR}_i(1), \mathrm{BR}_i(0)] = [0, 1/2],$$

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$$\begin{aligned} S_i^2 &= \mathrm{BR}_i([0,1/2]) = [\mathrm{BR}_i(1/2), \mathrm{BR}_i(0)] = [1/4, 1/2] \\ S_i^3 &= \mathrm{BR}_i([1/4,1/2]) = [\mathrm{BR}_i(1/2), \mathrm{BR}_i(1/4)] = [1/4, 3/8] \\ &\vdots \end{aligned}$$

$S_i^n = \mathrm{BR}_i(S_i^{n-1})$ converges to the fixed point of BR_i, which is the Nash equilibrium $s_i^* = 1/3$

References

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