Mixed Extension of a Game

(September 3, 2007)

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Ex : penalty kick, poker, rock-paper-scissors, tax inspections . . . image

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VNM preferences $\Rightarrow$ $\sigma = (\sigma_i)_{i \in N}$ is evaluated by player i with the expected utility function $U_i : \Sigma \to \mathbb{R}$

$$U_i(\sigma) = \sum_{s \in S} \sigma(s)\, u_i(s)$$

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Independent strategies $\Rightarrow$ $\sigma(s) = \prod_{i \in N} \sigma_i(s_i)$

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$$U_i(\sigma) = \; p\,q\,u_i(a,\,a) + p\,(1-q)\,u_i(a,\,b) + (1-p)\,q\,u_i(b,\,a) + (1-p)\,(1-q)\,u_i(b,\,b)$$

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that is, a profile of strategies $\sigma^* \in \Sigma$ such that

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \quad \forall\, \sigma_i \in \Sigma_i, \; \forall\, i \in N$$

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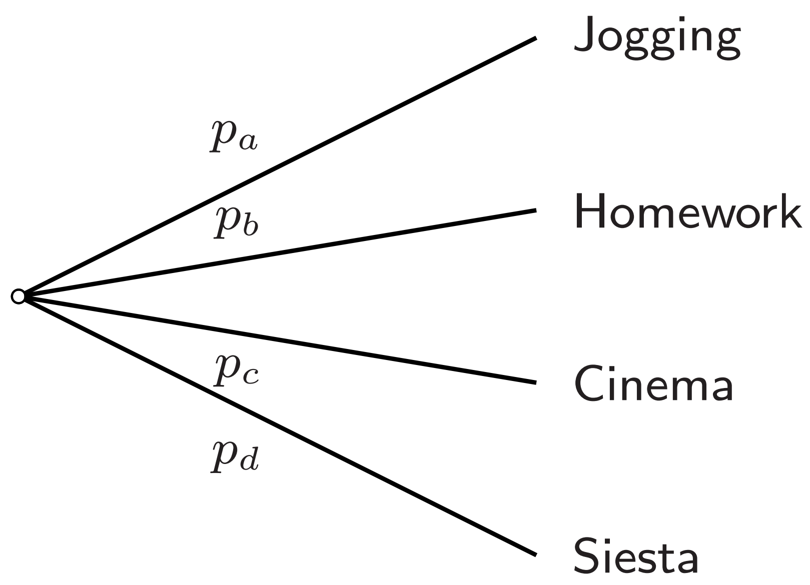

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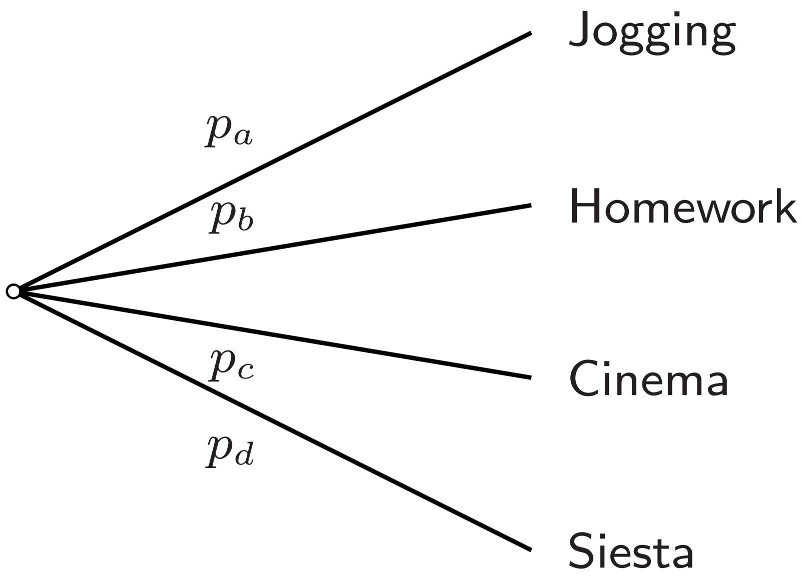

The decisionmaker chooses $\sigma_i = (\frac{5}{6}, \frac{1}{6}, 0, 0)$

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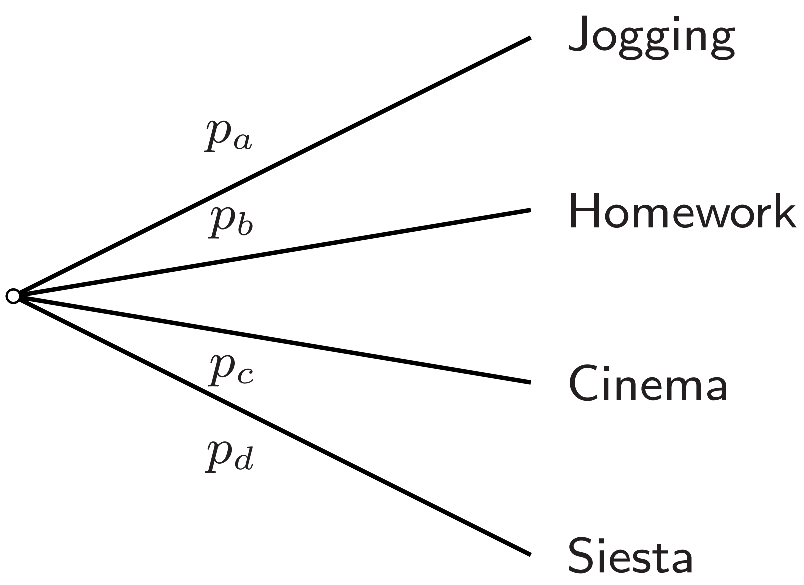

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$$\Rightarrow \text{supp}[\sigma_i] = \{a, b\}$$

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$\Rightarrow$ if the decision maker "throws a dice" to choose between a and b, then he should prefer a and b to all other actions and should be indifferent between a and b

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In particular, $\sigma_i \in \mathrm{BR}_i(\sigma_{-i})$ if and only if $s_i \in \mathrm{BR}_i(\sigma_{-i})$ for all $s_i \in \operatorname{supp}[\sigma_i]$. That is, $\max_{\sigma_i \in \Sigma_i} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$ $\square$

✍ Verify and explain why $\sigma_1 = (3/4, 0, 1/4)$ and $\sigma_2 = (0, 1/3, 2/3)$ is a Nash equilibrium (a point $\cdot$ can be any payoff)

	$G\ (0)$	$C\ (1/3)$	$D\ (2/3)$
$H\ (3/4)$	$(\cdot, 2)$	$(3, 3)$	$(1, 1)$
$M\ (0)$	$(\cdot, \cdot)$	$(0, \cdot)$	$(2, \cdot)$
$B\ (1/4)$	$(\cdot, 4)$	$(5, 1)$	$(0, 7)$

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Proof. Directly from the proposition on the existence of a symmetric pure strategy equilibrium □

Examples

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Prisoners dilemma.

	D	C
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(D, D) is the unique Nash equilibrium because D strictly dominates C for both players

Coordination game.

	a	b	
a	$(2,2)$	$(0,0)$	p
b	$(0,0)$	$(1,1)$	$1-p$
	q	$1-q$	

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If $p \in \{0,1\}$ we get the two pure strategy NE (a,a) and (b,b)

If $0 < p < 1$ then player 1 should be indifferent

$$a \xrightarrow{1} u_1(a,\sigma_2) = 2q + 0(1-q) = 2q$$

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Coordination game.

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a	$(2,2)$	$(0,0)$	p
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Player 2 plays $\sigma_2(a) = q = 1/3$ if he is also indifferent. By symmetry, $p = \frac{1}{3}$

$\Rightarrow$ Non degenerate mixed strategy NE:

$$\sigma = \left(\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \right)$$

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$\Rightarrow$ 3 NE, including 2 in pure strategies

Best responses correspondences:

$$\mathrm{BR}_i(\sigma_j) = \begin{cases} \{0\} & \text{if } \sigma_j(a) < \frac{1}{3} \\ [0,1] & \text{if } \sigma_j(a) = \frac{1}{3} \\ \{1\} & \text{if } \sigma_j(a) > \frac{1}{3} \end{cases}, \quad i = 1, 2,\ j \neq i$$

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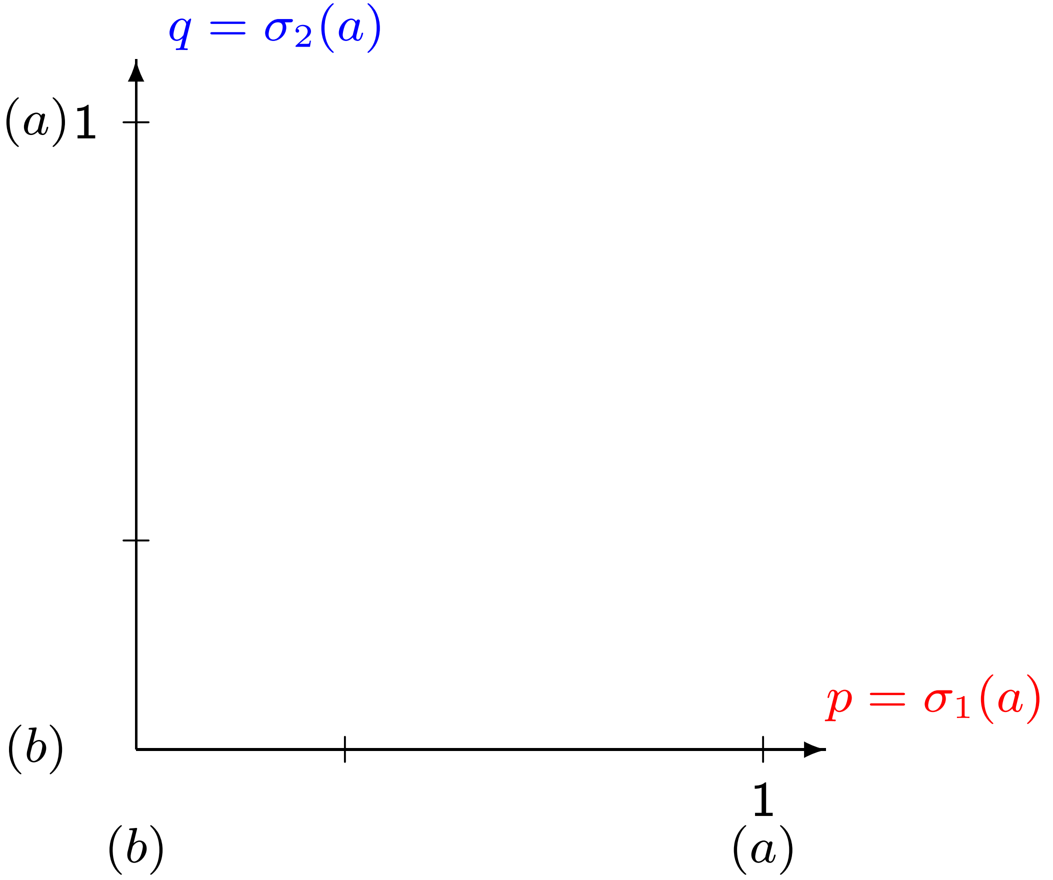

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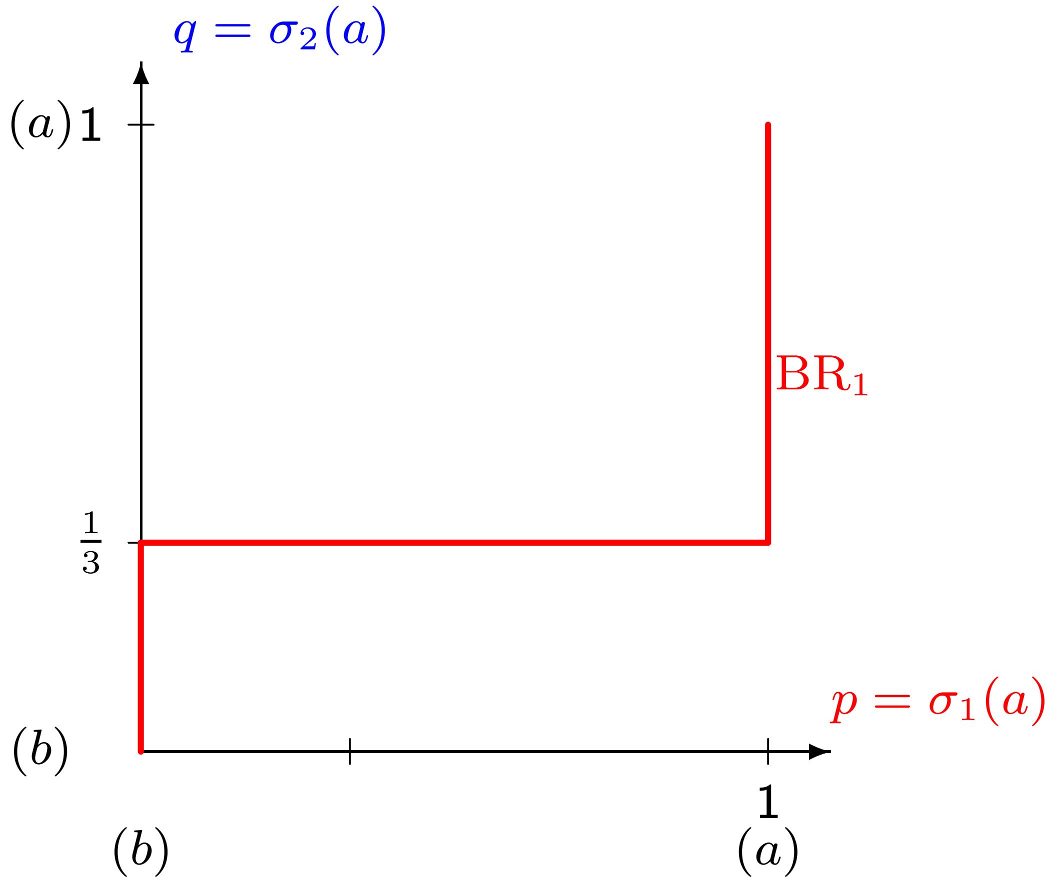

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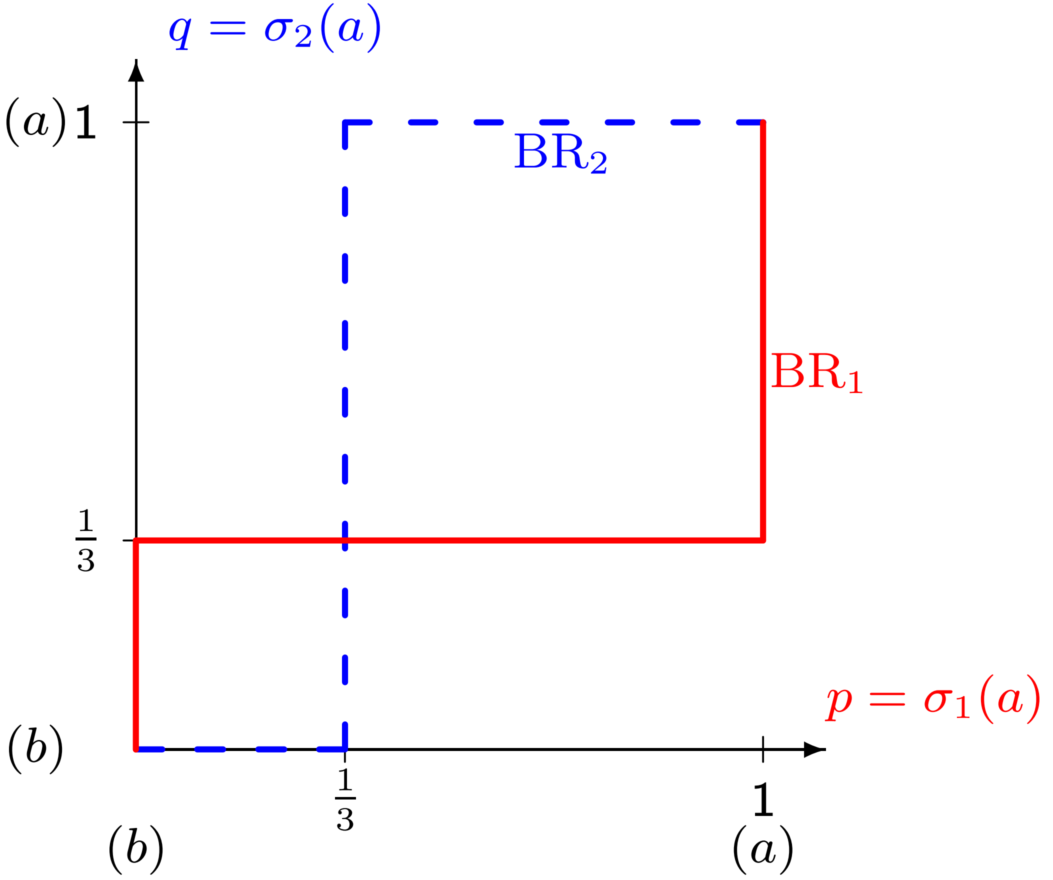

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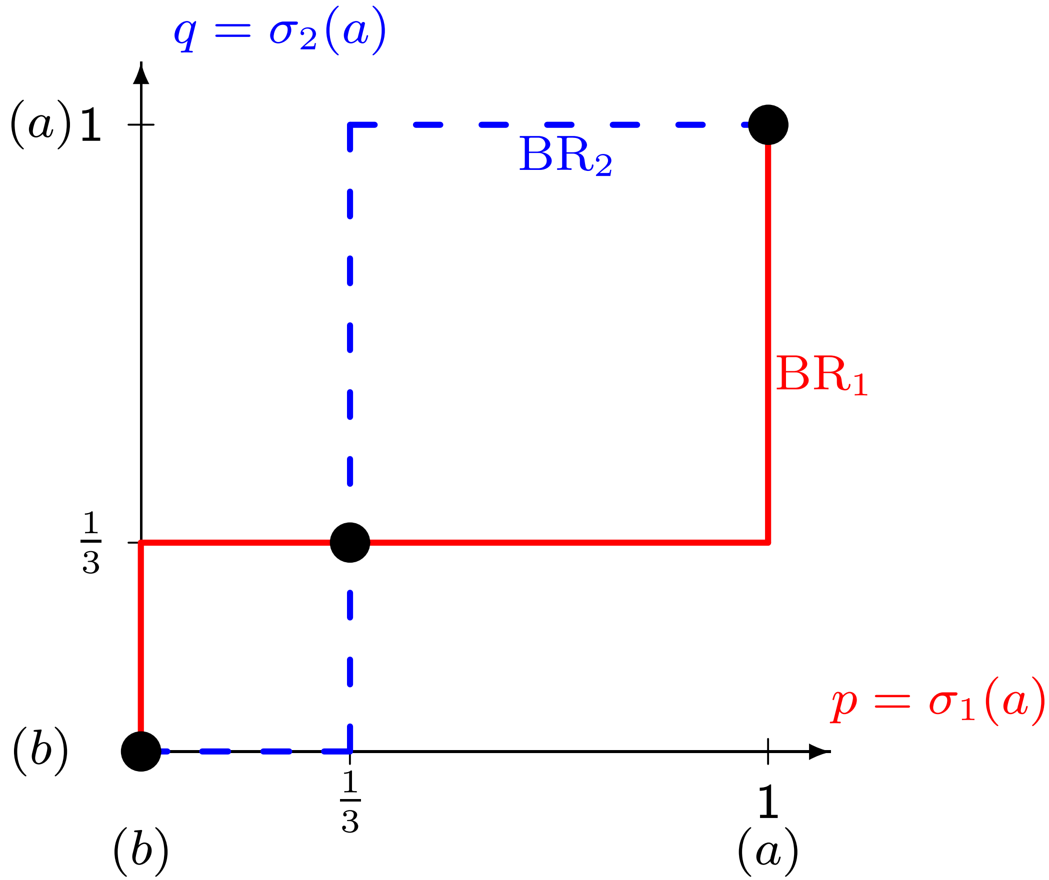

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$$a \xrightarrow{2} 2p$$

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Battle of sexes.

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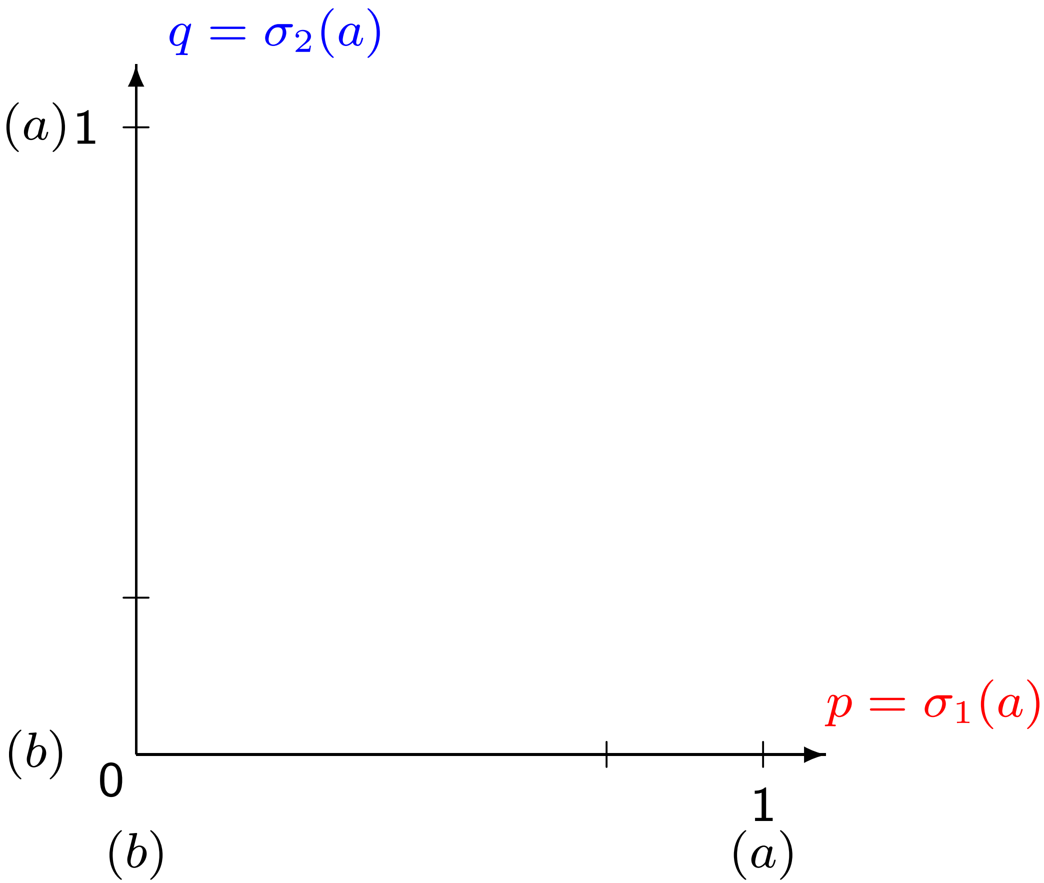

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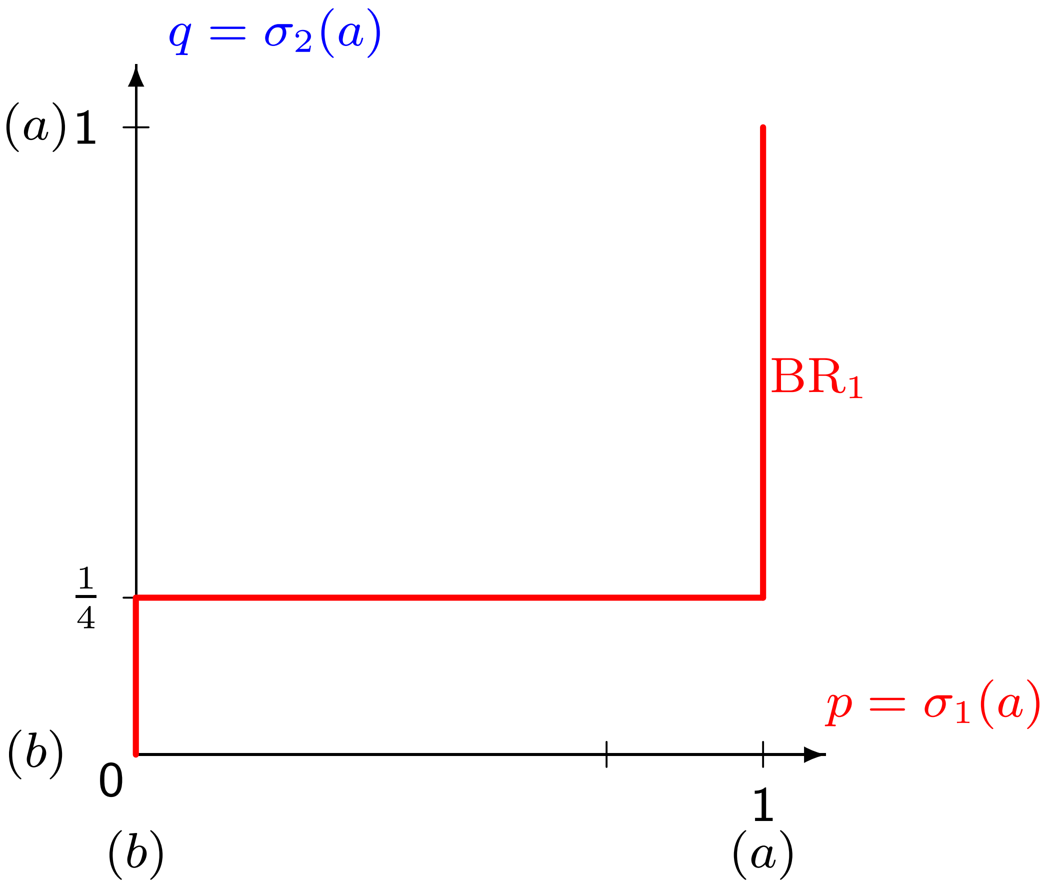

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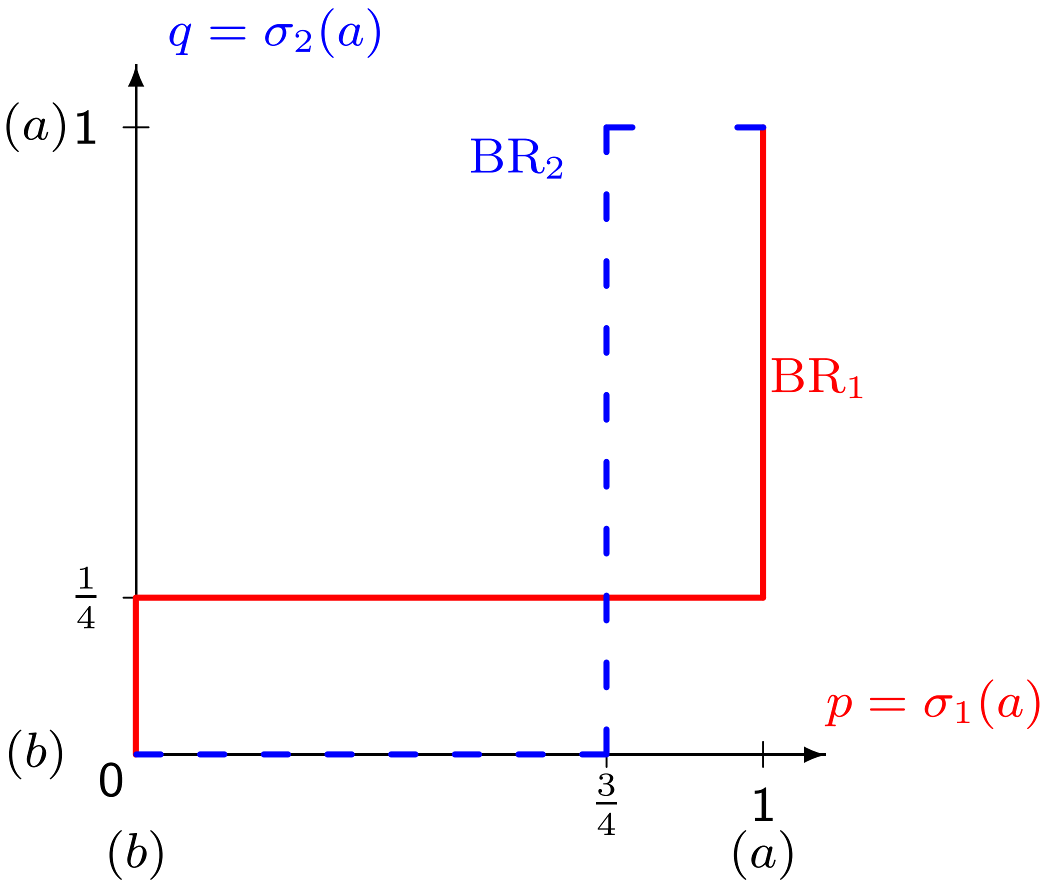

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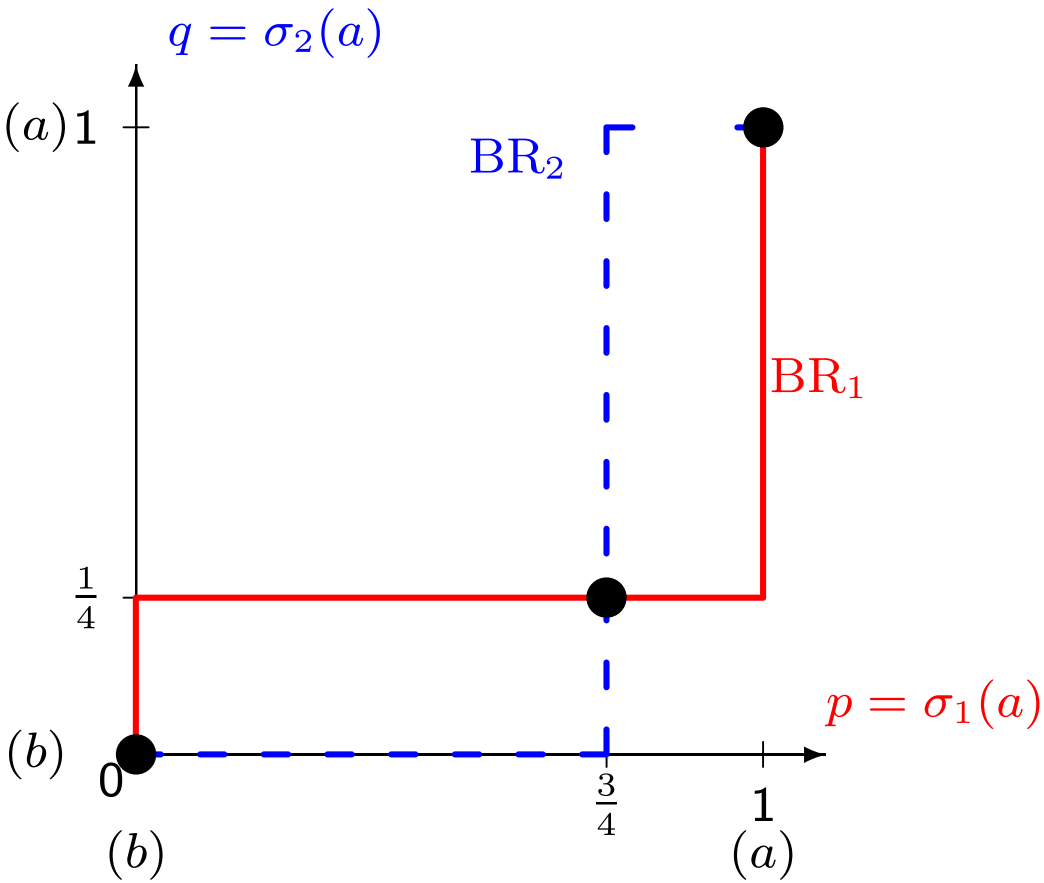

Chicken game.

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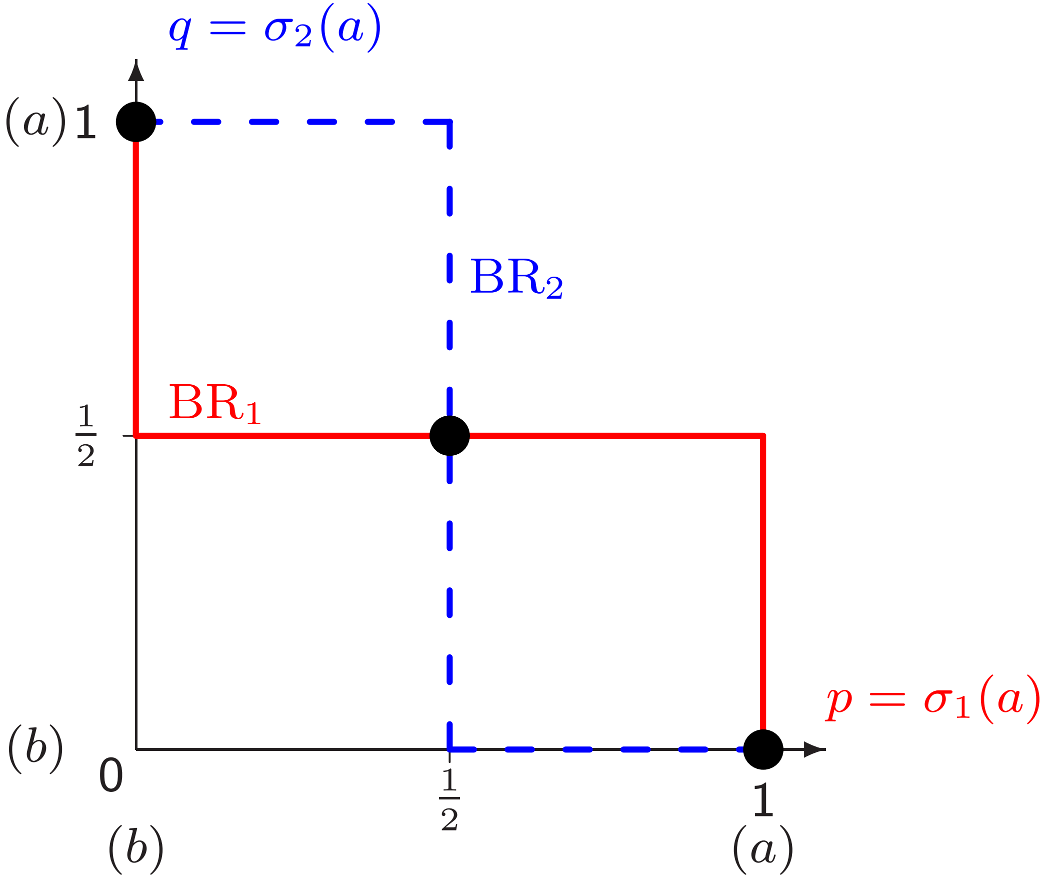

Stag hunt.

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$$
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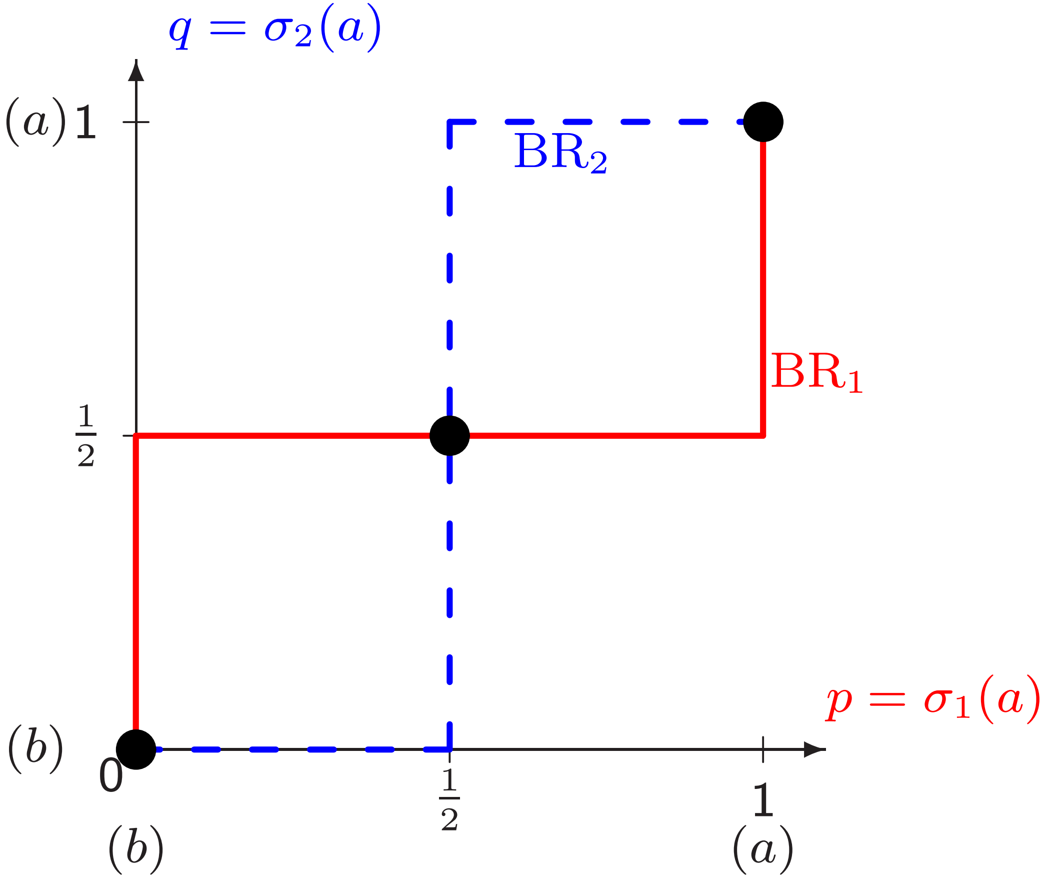

Matching pennies.

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	G	D	
G	$(-1,1)$	$(1,-1)$	p
D	$(1,-1)$	$(-1,1)$	$1-p$
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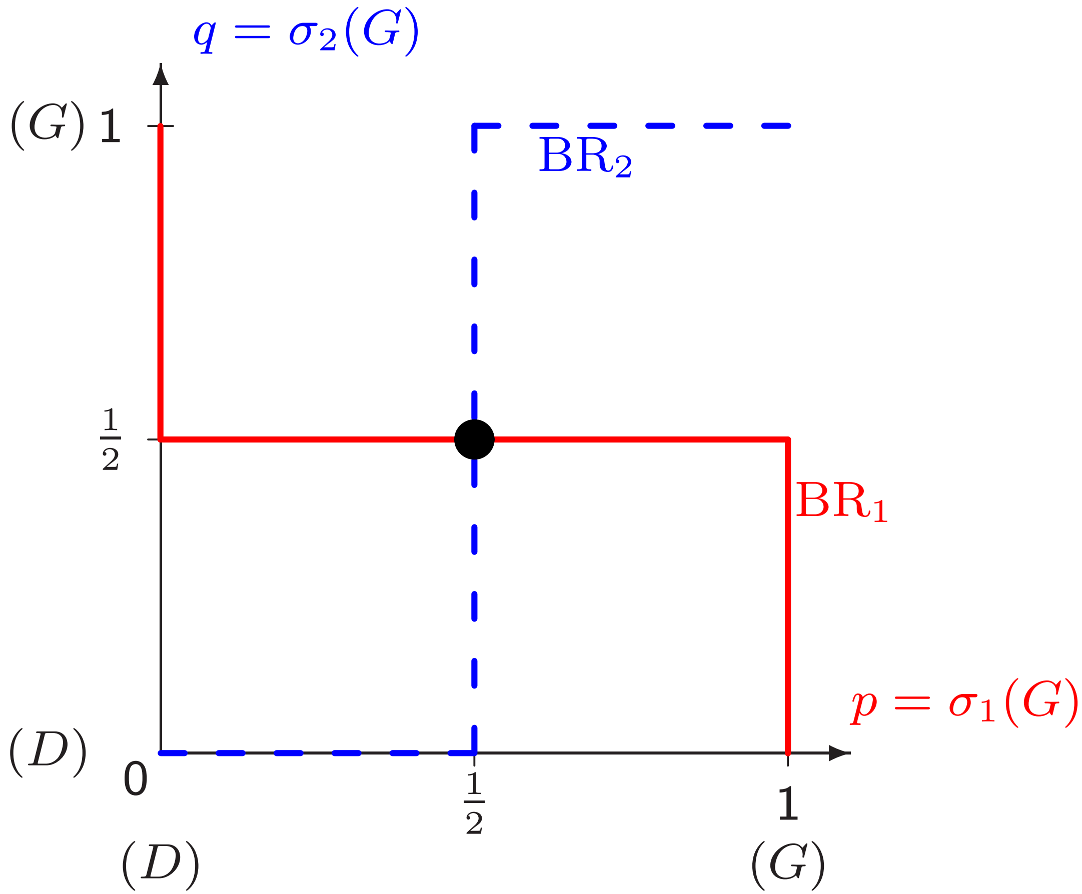

Paper, Rock, Scissors.

	F	P	C	
F	$(0,0)$	$(1,-1)$	$(-1,1)$	a
P	$(-1,1)$	$(0,0)$	$(1,-1)$	b
C	$(1,-1)$	$(-1,1)$	$(0,0)$	c
	p	q	r	

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$a = b = c = 1/3$ and $p = q = r = 1/3$

Reporting a Crime

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- ➢ audience inhibition
- ➢ social influence

☛ All these factors raise the expected cost and/or reduce the expected benefit of a person's intervening


"RAPED. TORTURED. KILLED. HERE. IN NEW YORK. OUTSIDE HER OWN APARTMENT BUILDING.
"ALMOST FORTY NEIGHBORS HEARD SCREAMS. NOBODY DID ANYTHING. NOBODY CALLED COPS. SOME OF THEM EVEN WATCHED. DO YOU UNDERSTAND?

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But coordination to such an asymmetric equilibrium is difficult without communication or convention

Symmetric equilibrium here: in (non-degenerated) mixed strategies

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✔ $\mathrm{Pr}(\text{at least one person calls}) = 1 - (1-q)^n = 1 - (c/v)^{n/n-1}$ decreases with n !

Reporting a crime when the witnesses are heterogeneous.

✍ Consider a variant of the model in which n_1 witnesses incur the cost c_1 to report the crime, and n_2 witnesses incur the cost c_2, where $0 < c_1 < v$, $0 < c_2 < v$, and $n_1 + n_2 = n$. Show that if c_1 and c_2 are sufficiently close, then the game has a mixed strategy Nash equilibrium in which every witness's strategy assigns positive probabilities to both reporting (calling the police) and not reporting.

Finding all Nash Equilibria

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	A	B	C
a	$3, 2$	$1, 1$	$0, 1.5$
b	$0, 0$	$2, 3$	$1, 2$

Figure 1: A variant of the battle of sexes

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We find $(\sigma_1, \sigma_2) = ((4/5, 1/5),\ (1/4, 0, 3/4))$

Prudent / Maxmin Strategy

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2-player finite games $\langle \{1, 2\}, (S_1, S_2), (u_1, u_2) \rangle$ image

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Payoff guaranteed by strategy $s_1 \in S_1$ of player 1 = worst payoff that player 1 can get by playing action s_1 :

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A maxmin action is an action that maximizes the payoff that the player can guarantee $\Rightarrow$ "maxminimizing" action

Definition. An action $s_1^* \in S_1$ is a **maxmin** or **maxminimizing** action for player 1 if

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☞ A maxmin strategy profile is not necessarily a Nash equilibrium

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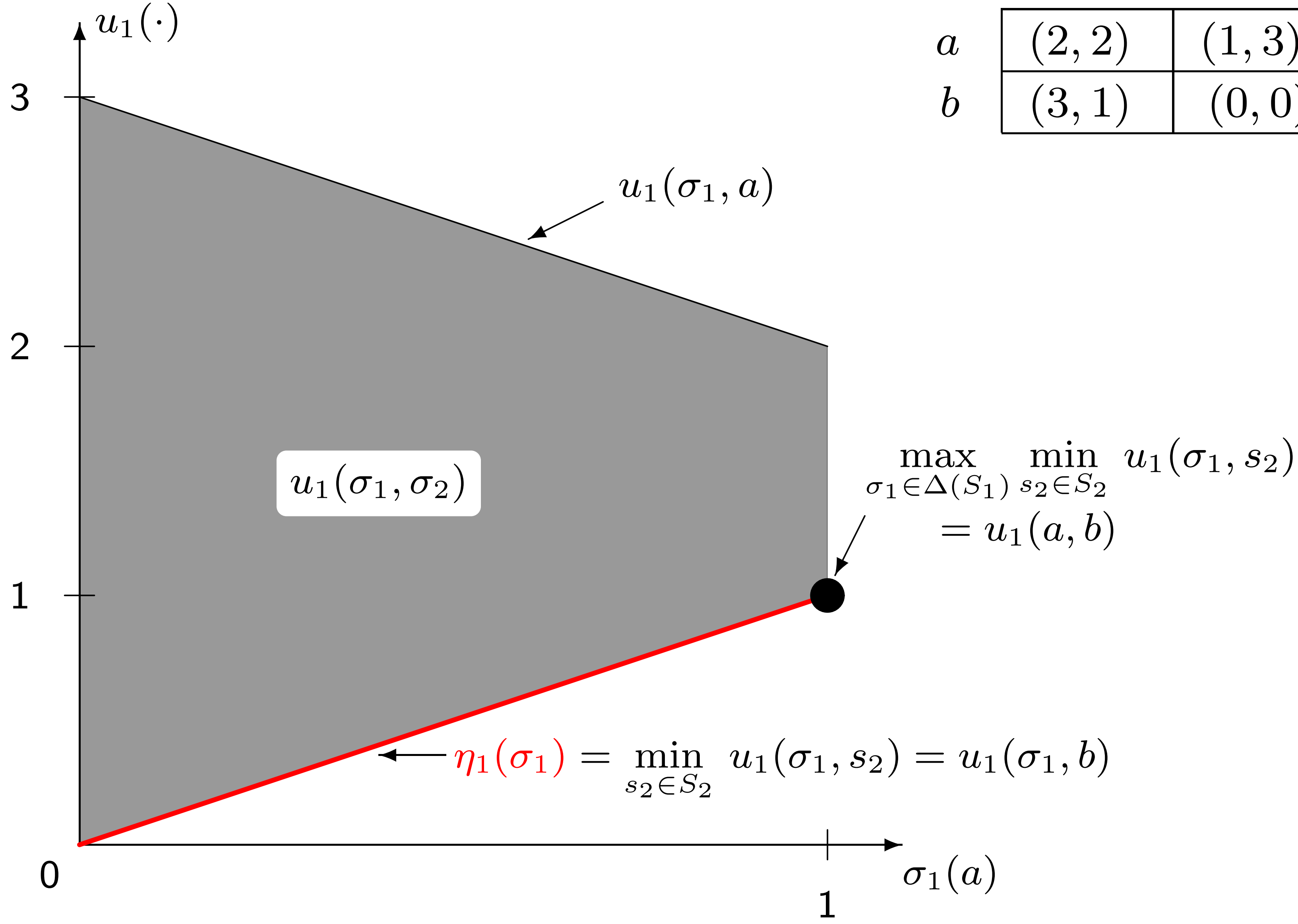

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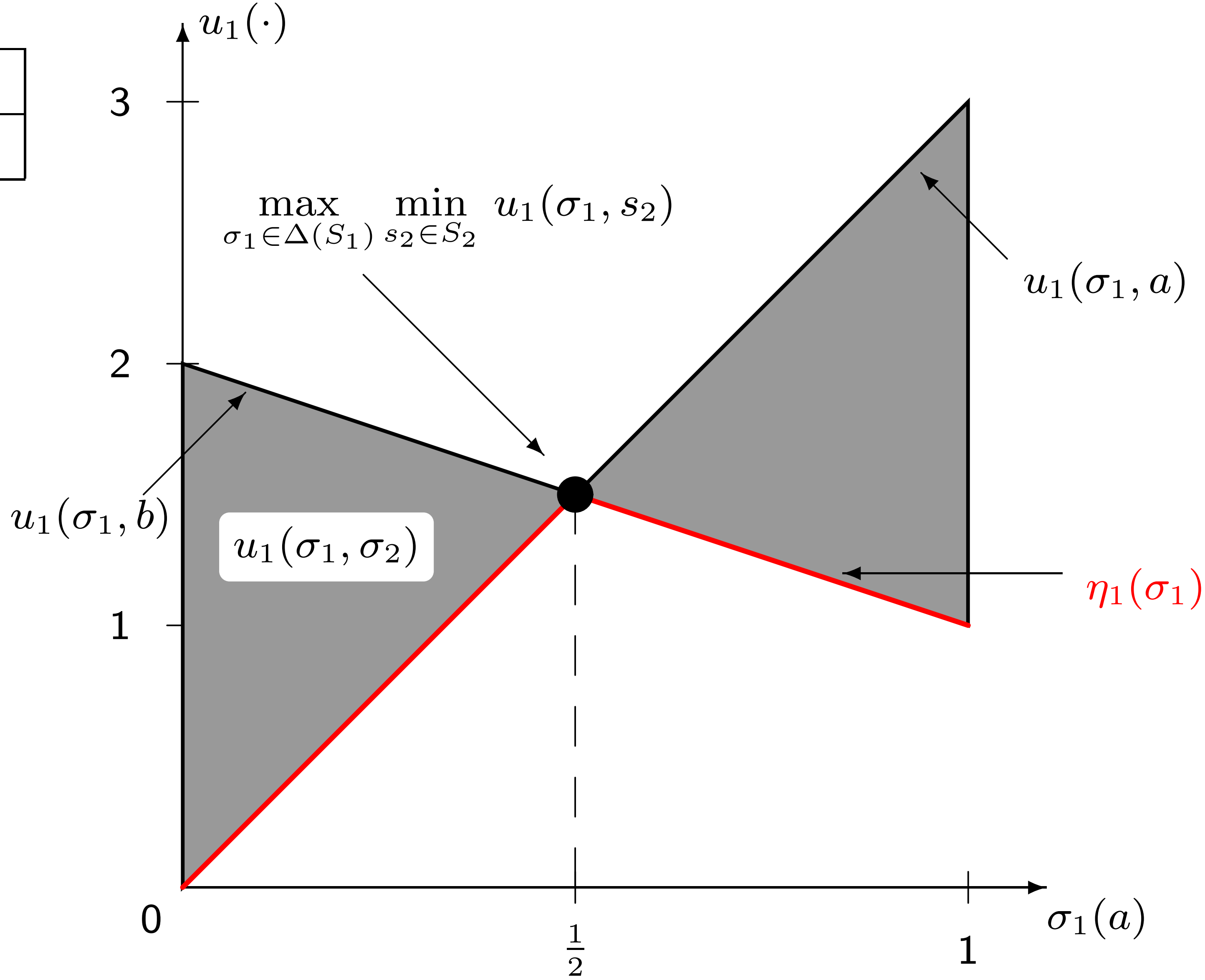

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Examples : matching pennies, rock-paper-scissors, chess

✍ Find other examples of 0-sum games

Theorem. *In a finite zero-sum game, (σ_1^*, σ_2^*) is a Nash equilibrium if and only if (σ_1^*, σ_2^*) is a maxmin strategy profile. In addition, we have*

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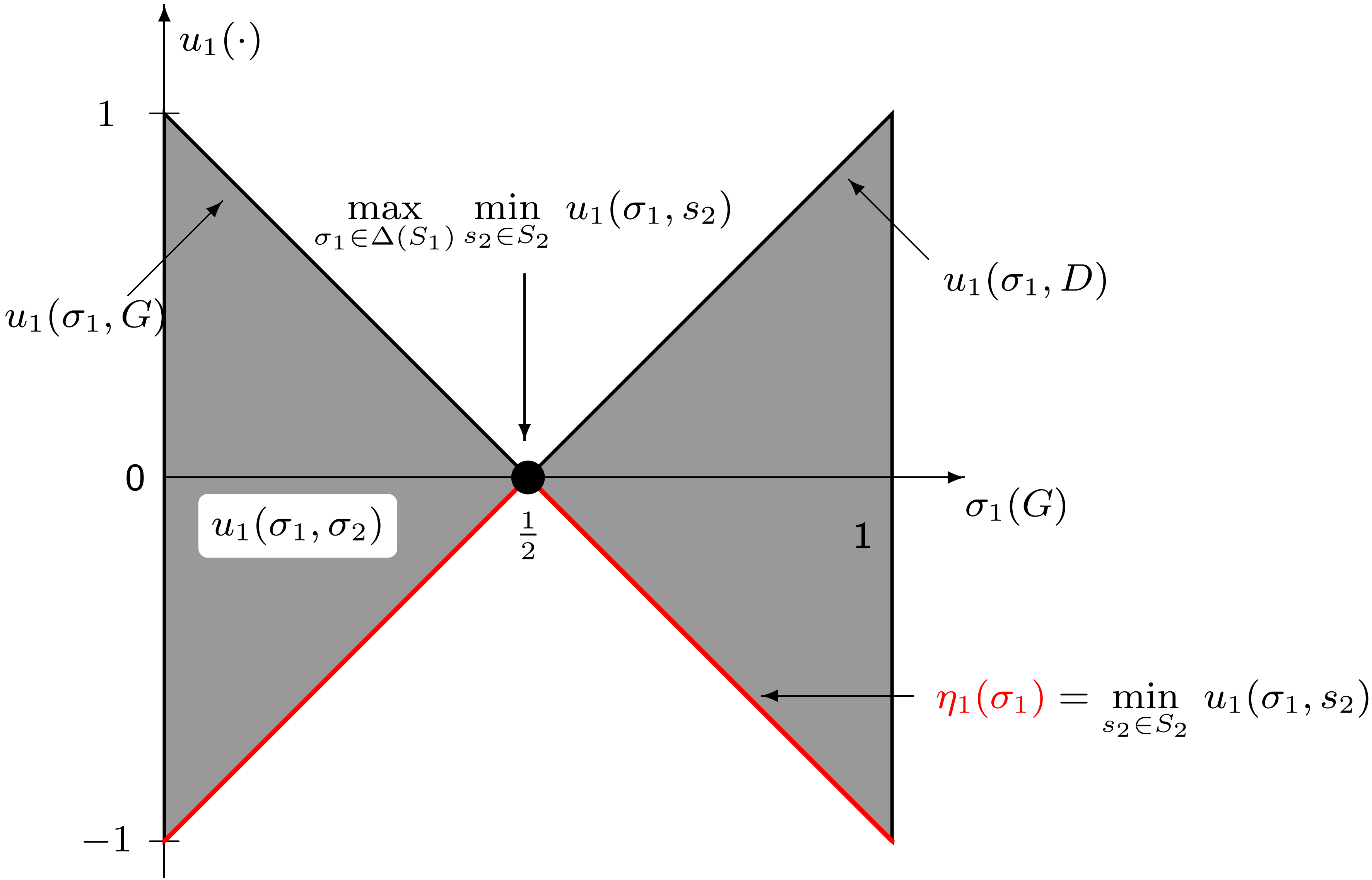
Example. In matching pennies the maxmin strategy of player 1 is indeed equivalent to his equilibrium strategy $\sigma_1^*(G) = 1/2$



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Proof. Directly from the fact that in zero-sum games

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In non zero-sum games, consider

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✍ Explain why in a finite symmetric zero-sum game players' payoff is zero in every Nash equilibrium

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  Common knowledge of rationality

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Example.

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Remark. If s_i is strictly (weakly) dominated then every mixed strategy putting strictly positive probability on s_i is strictly (weakly) dominated

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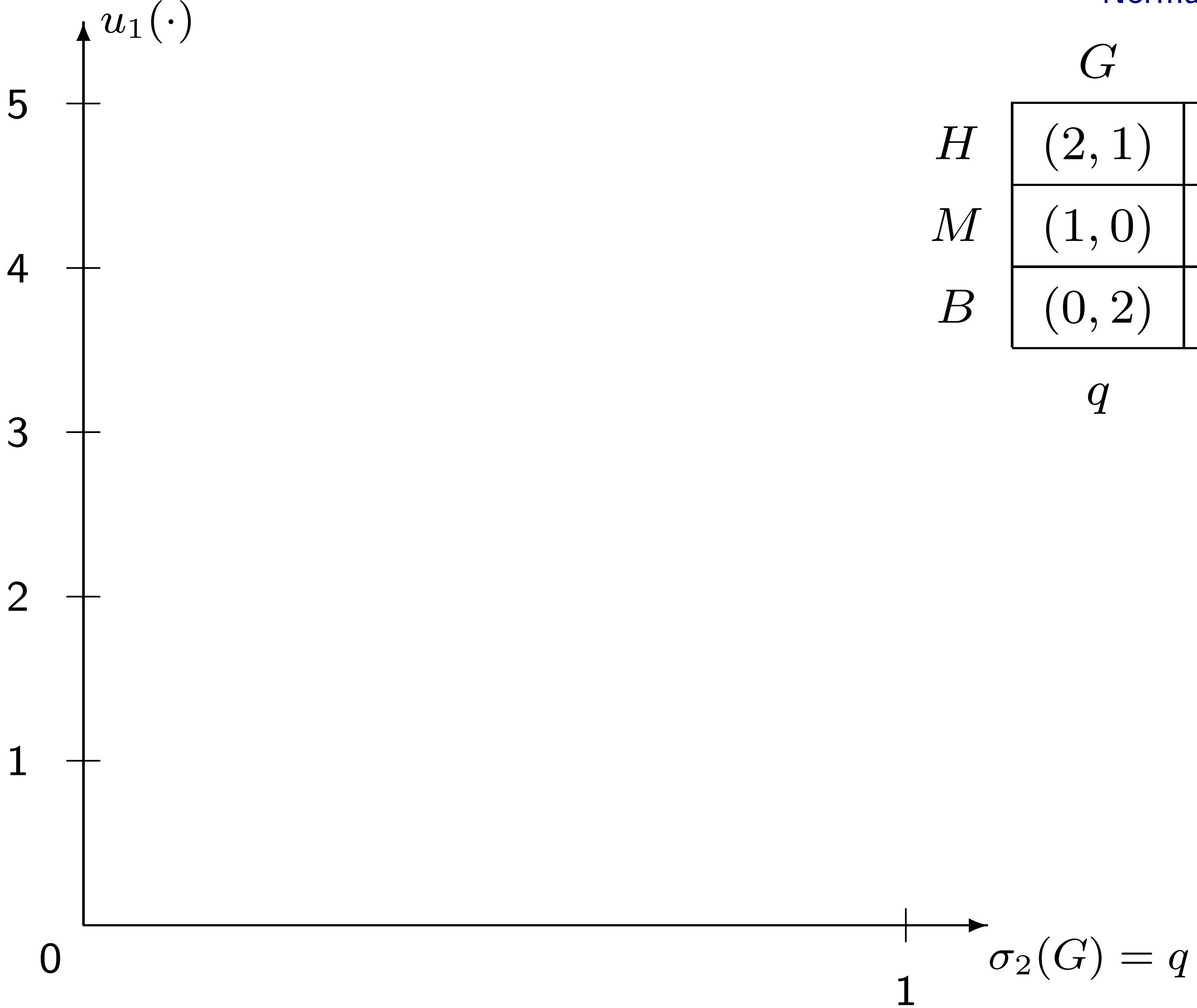

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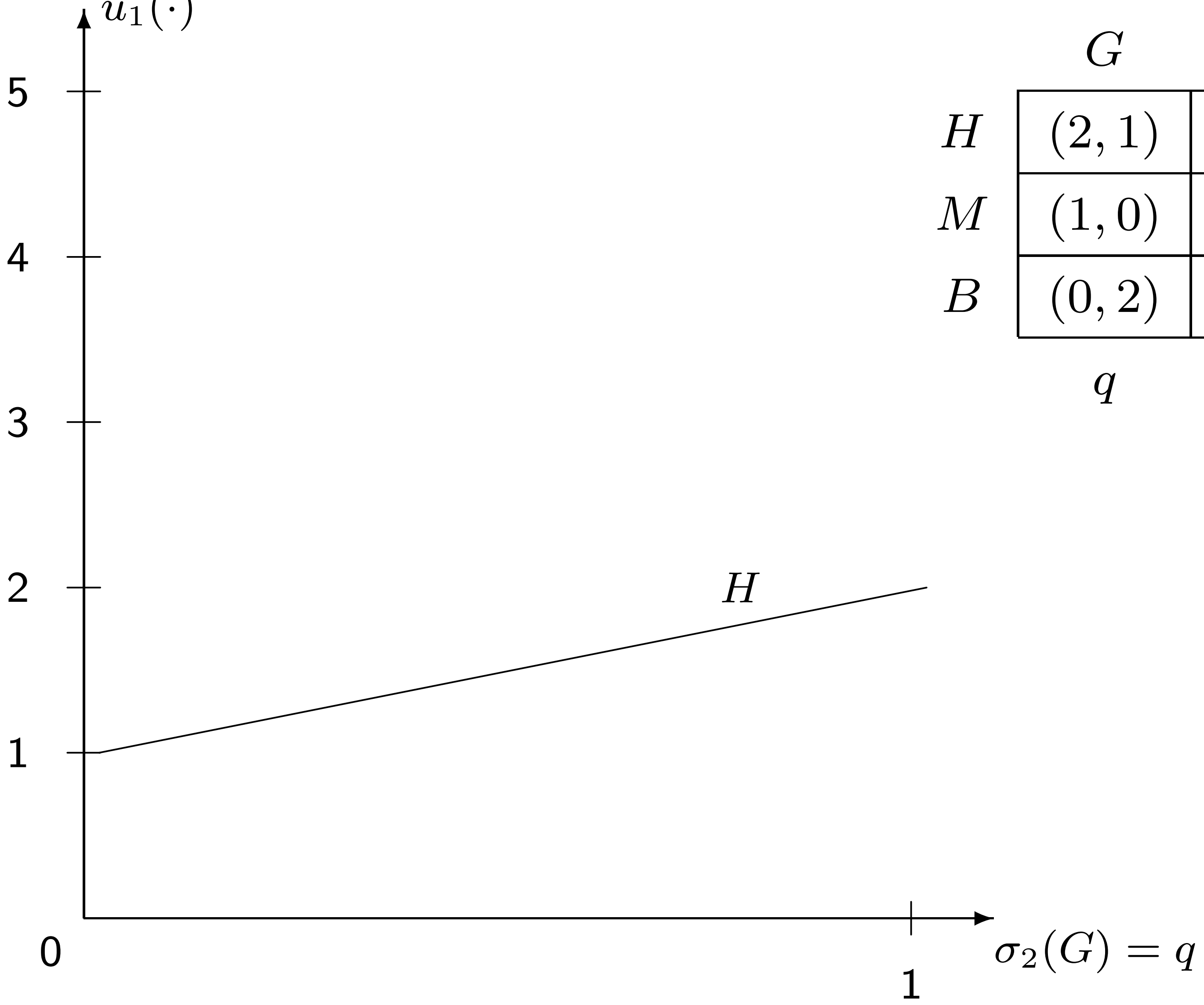

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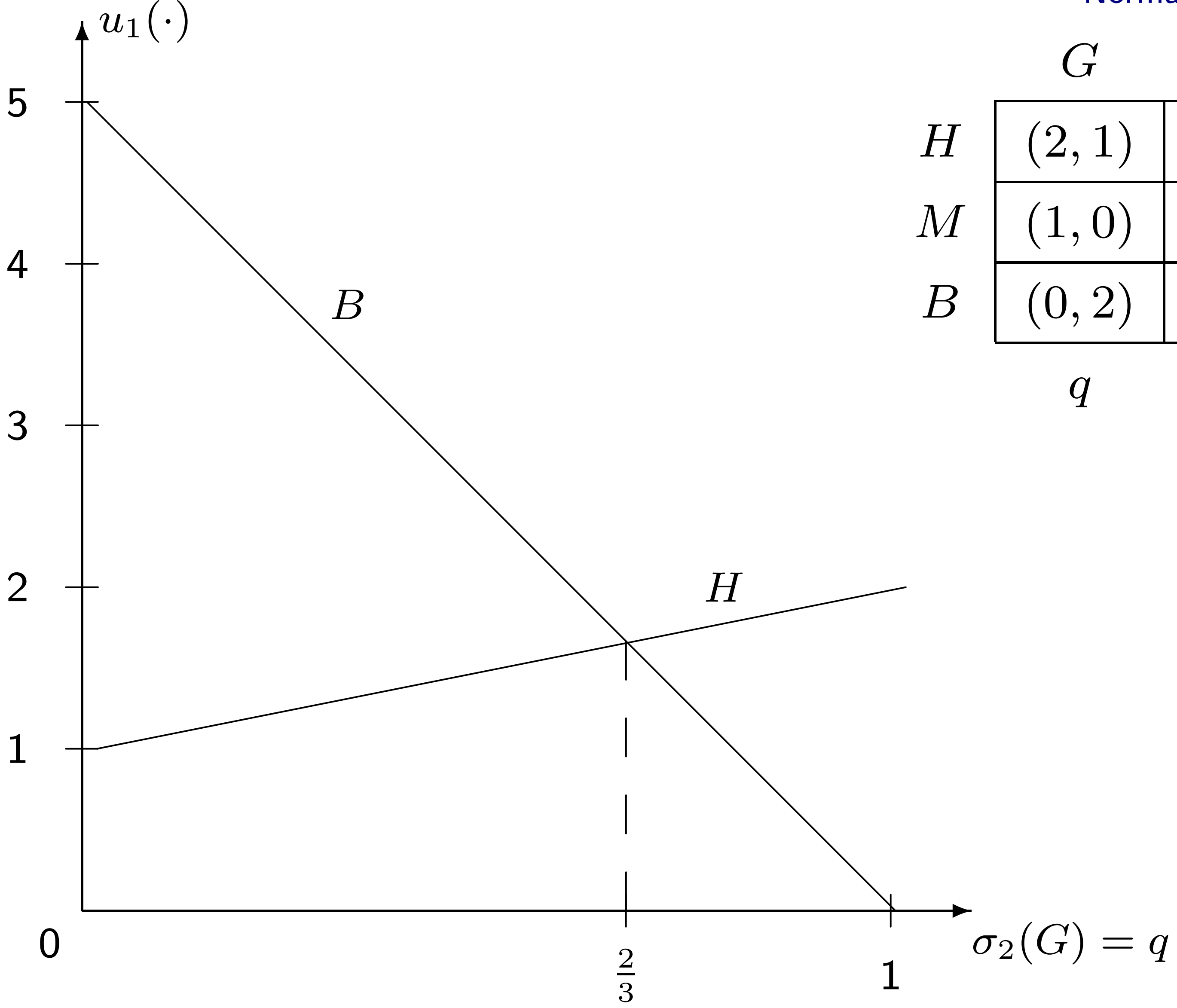

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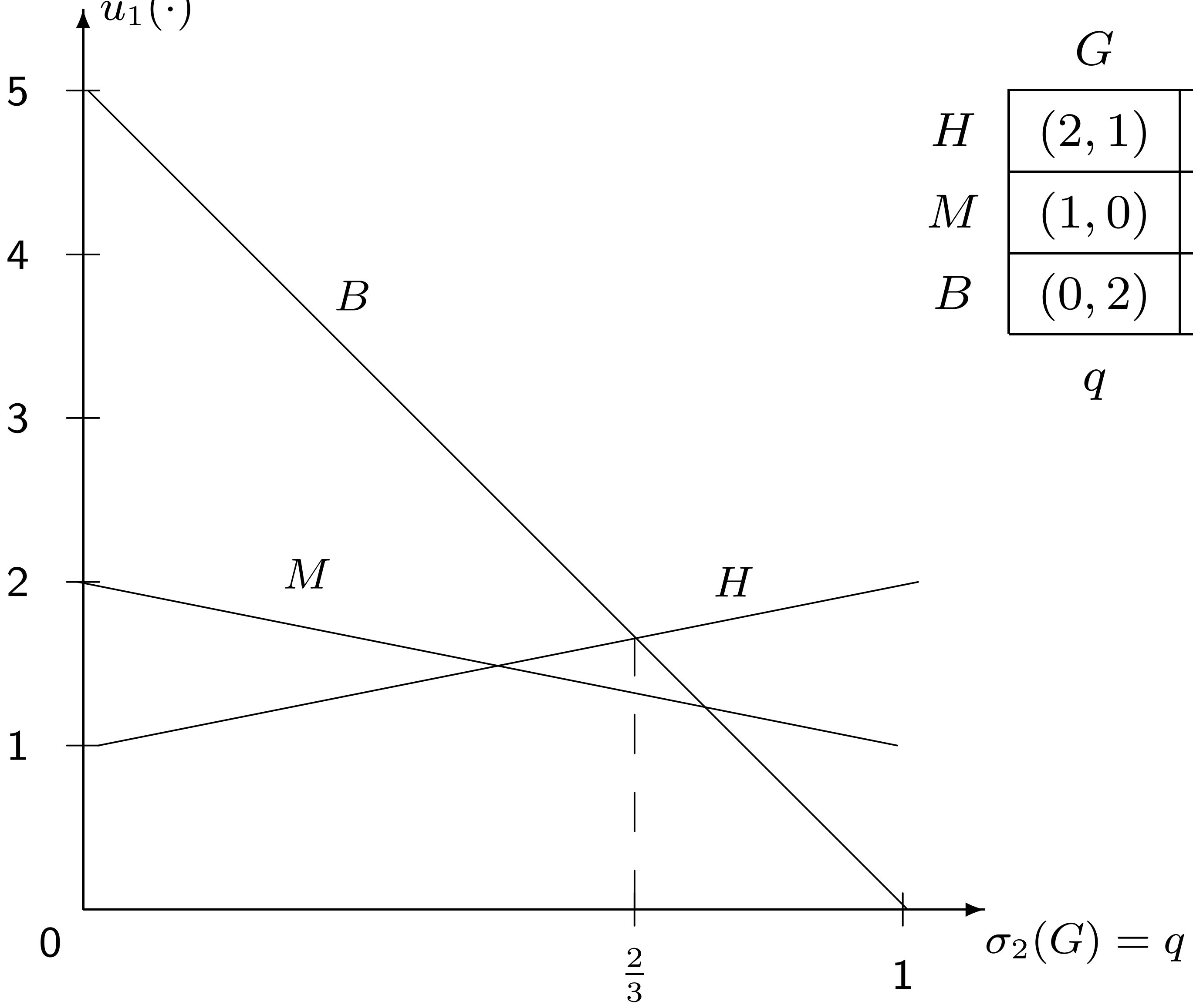

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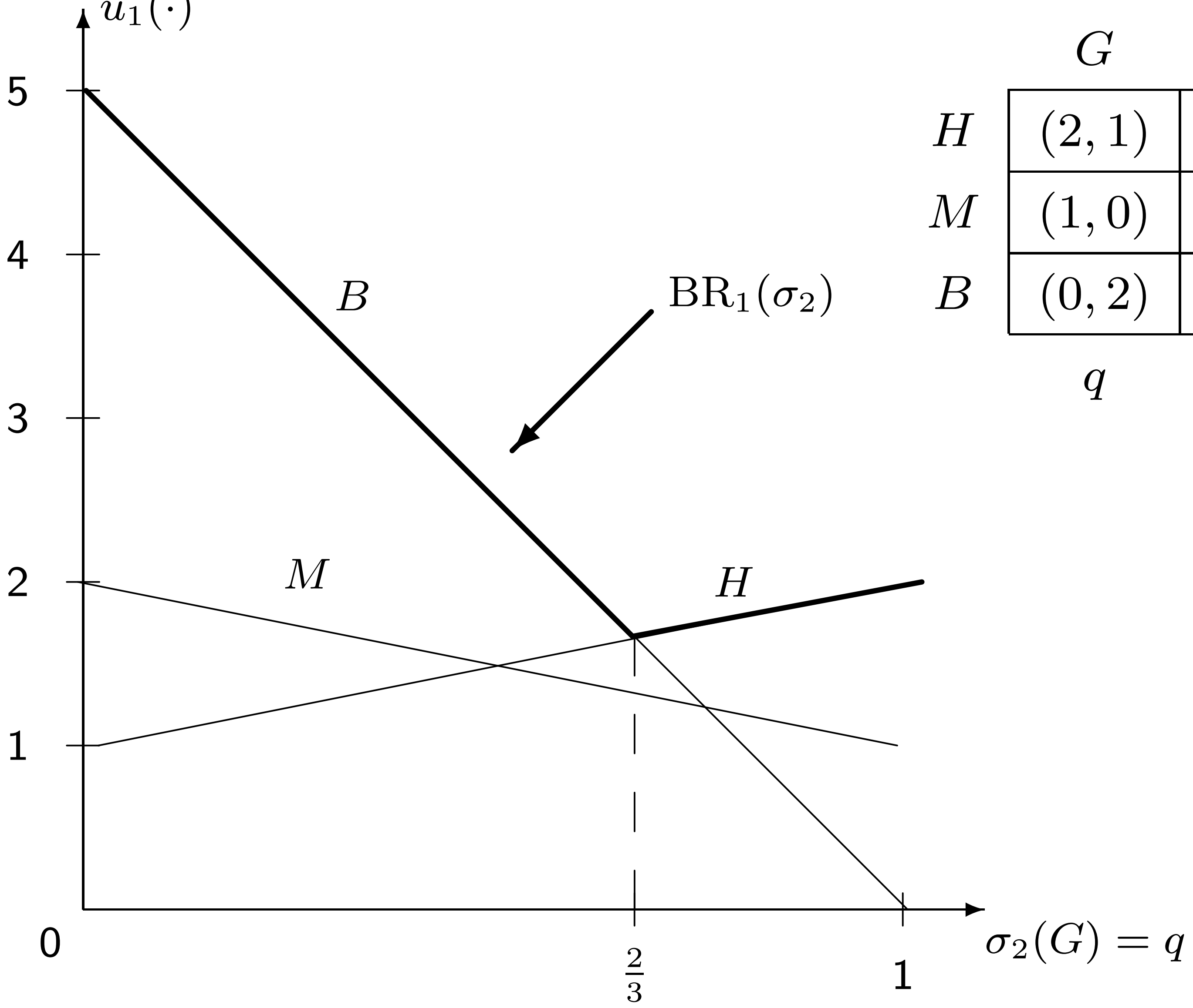

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M is never a best response, so it is strictly dominated (✍ by which strategy?)

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Definition. A set of strategy profiles $S^* \subseteq S$ **survives iterated elimination of strictly dominated strategies** if there is a sequence $(S^k)_{k=1}^K$, where $S^k = (S_i^k)_{i \in N}$ for every k, such that

- $S^0 = S$ and $S^K = S^*$
- $S^{k+1} \subseteq S^k$ for every k
- if $s_i \in S_i^k$ but $s_i \notin S_i^{k+1}$ then s_i is strictly dominated in the reduced game $G^k = \langle N, (S_i^k)_i, (u_i)_i \rangle$
- no action is strictly dominated in the reduced game $G^K = \langle N, (S_i^K)_i, (u_i)_i \rangle$

Example.

	G	D
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$\Rightarrow$ the ordering of elimination does not influence the final outcome

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Proposition. *Any action played with positive probability at a Nash equilibrium survives iterated elimination of strictly dominated strategies. This is not necessarily true for iterative elimination of weakly dominated strategies. However, after iteratively eliminating weakly dominated strategies, there is always at least one Nash equilibrium of the original game that survived*

Example. Cournot duopoly with $\theta_1 = \theta_2 = -1$:

$$u_i(s_1, s_2) = s_i(1 - s_1 - s_2) \qquad \mathrm{BR}_i(s_j) = \left\{ \frac{1 - s_j}{2} \right\}$$

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$S_i^n = \mathrm{BR}_i(S_i^{n-1})$ converges to the fixed point of BR_i, which is the Nash equilibrium $s_i^* = 1/3$

References

von Neumann, J. (1928): "Zur Theories der Gesellschaftsspiele," Math. Ann., 100, 295–320.