Incomplete Information and Bayesian Games

Outline

(October 27, 2008)

1/

- Information structure, knowledge and common knowledge, beliefs
- Bayesian game and equilibrium
- Applications
  - No bet/trade theorems
  - Reinterpretation of mixed strategies
  - Correlation and communication

2/

Implicit assumption in games (normal and extensive forms):

Every player perfectly knows the game

However, in many economic situations, *information* is *imperfect* and *asymmetric*:

☞ Policymakers: state of the economy, consumers and firms' preferences

☞ Firms: costs, level of demand, other firms' R&D output

☞ Negotiators: others' valuations and costs, ...

☞ Bidders: value of the object, other bidders' valuations

☞ Shareholders: value of the firm

☞ Contractual relationships: The principal (insurer, employer, regulator, ...) does not know the "type" of the agent(s)

Information System

- ➢ Set of **states of the world**: Ω

  $\omega \in \Omega$: complete description of the situation (players' preferences and information)

- ➢ **Information function** of player i:

$$P_i : \Omega \to 2^{\Omega}$$

3/ Assumptions:

$\omega \in P_i(\omega)$ for every $\omega \in \Omega$: *correct* ("truth axiom")

$\omega' \in P_i(\omega) \Rightarrow P_i(\omega') = P_i(\omega)$: *partitional*

➥ Partition $\mathcal{P}_i = \{P_i(\omega) : \omega \in \Omega\}$ of player i

Information set of player i at ω: $P_i(\omega) =$ element of $\mathcal{P}_i$ containing ω

Every player knows others' partitions (otherwise ω is not a **complete** description of the situation)

Examples

$\Omega = \{00, 01, 02, \ldots, 97, 98, 99\}$ and the agent can only read the first digit:

4/

$$\begin{array}{lclclcl} P_i(00) & = & \ldots & = & P_i(09) & = & \{00, 01, \ldots, 09\} \\ \vdots & & \vdots & & & & \vdots \\ P_i(k0) & = & \ldots & = & P_i(k9) & = & \{k0, k1, \ldots, k9\} \\ \vdots & & \vdots & & & & \vdots \\ P_i(90) & = & \ldots & = & P_i(99) & = & \{90, 91, \ldots, 99\} \end{array}$$

$\Rightarrow$ Partition $\mathcal{P}_i = \{\{00, \ldots, 09\}, \ldots, \{90, \ldots, 99\}\}$

Correct ($\omega \in P_i(\omega)$ for every $\omega \in \Omega$)

$\Omega = \{00, 01, 02, \ldots, 97, 98, 99\}$ and the agent can read both digits but he reads it in the wrong way round:

$$P_i(kl) = \{lk\}$$

$\Rightarrow$ partition but $\omega \notin P_i(\omega)$ (*errors*)

5/ $\Omega = \{B, M\}$ and the agent only remembers good news:

$$P_i(B) = \{B\} \quad P_i(M) = \{B, M\}$$

$\Rightarrow$ $\omega \in P_i(\omega)$ for every ω: correct information but not partitional:

$B \in P_i(M)$ but $P_i(B) \neq P_i(M)$ (*imperfect introspection*)

Player i is **more informed** than player j if partition $\mathcal{P}_i$ is finer than $\mathcal{P}_j$, i.e.
$P_i(\omega) \subseteq P_j(\omega) \; \forall \; \omega \in \Omega$

Examples

Coin flip, only player 1 observes the outcome:

$$\Omega = \{H, T\} \quad \mathcal{P}_1 = \{\{H\}, \{T\}\} \quad \mathcal{P}_2 = \{\{H, T\}\}$$

6/ ☞ Player 1 if more informed than player 2

Player 1 does not know whether player 2 has cheated:

$$\Omega = \{H, H^C, T, T^C\} \quad \mathcal{P}_1 = \{\{H, H^C\}, \{T, T^C\}\} \quad \mathcal{P}_2 = \{\{H, T\}, \{H^C\}, \{T^C\}\}$$

☞ No player is more informed than the other

Individual Knowledge

Knowledge operator : $K_i : 2^\Omega \to 2^\Omega$

$K_iE \quad = \{\omega \in \Omega : P_i(\omega) \subseteq E\}$

= set of states in which player i knows **that** the event E is realized

7/ $W_iE \quad = K_iE \cup K_i\neg E$

= set of states in which player i knows **whether** the event E is realized

Properties of the knowledge operator K_i.

$K_i\Omega = \Omega$ (*necessitation*): an agent always knows that the universal event Ω is realized. No unforeseen contingencies

$K_i(E \cap F) = K_iE \cap K_iF$ (axiom of *deductive closure*): an agent knows E and F iff he knows E and he knows F ($\Rightarrow$ logical omniscience: $E \subseteq F \Rightarrow K_iE \subseteq K_iF$)

8/ $K_iE \subseteq E$ (*truth axiom*): what the agent knows is true. Allow to distinguish the concept of *knowledge* from the concept of *belief*

$K_iE \subseteq K_i^2E$ (*positive introspection axiom*): if an agent knows E, then he knows that he knows E

$\neg K_iE \subseteq K_i\neg K_iE$ (*negative introspection axiom*): if an agent does not know E, then he knows that he does not know E (most restrictive axiom)

Example. $\Omega = \{1, 2, 3, 4\}$ $\mathcal{P}_1 = \{\{1\}, \{2\}, \{3, 4\}\}$ $\mathcal{P}_2 = \{\{1, 2\}, \{3, 4\}\}$

$E = \{3\} \Rightarrow K_1E = K_2E = \emptyset$: nobody knows E

$E = \{1, 3\} \Rightarrow K_1E = \{1\},\ K_2E = \emptyset,\ K_1\neg E = \{2\}$

$\Rightarrow W_1E = \{1, 2\},\ K_2W_1E = \{1, 2\},\ K_2\neg W_1E = \{3, 4\},\ W_2W_1E = \Omega$

$\Rightarrow E$ is *private knowledge* for player 1 at $\omega = 1$

9/ and player 2 always knows whether player 1 knows E

If $\mathcal{P}_2 = \{\{1, 2, 3, 4\}\}$ then $K_2W_1E = \emptyset,\ K_2\neg W_1E = \emptyset,\ W_2W_1E = \emptyset$

i.e., E is *private* and *secret* knowledge for player 1 at $\omega = 1$

(player 2 never knows whether player 1 knows E)

Interactive Knowledge

Mutual/shared Knowledge:

$KE = \bigcap_{i \in N} K_iE$

= set of states in which all players know E

10/ **Mutual knowledge at order** k:

$K^kE = \underbrace{K \cdots K}_{k \text{ times}} E$

= set of states in which everybody knows that everybody knows ... [k times] that E is realized

Common Knowledge (Lewis, 1969; Aumann, 1976):

CKE $= K^\infty E$

$=$ set of states in which everybody knows that everybody knows

... [at infinity] that E is realized

$= \{\omega \in \Omega : M(\omega) \subseteq E\}$

where $M(\omega)$ is the cell of the *common knowledge partition* ("*Meet*"),

11/ $\mathcal{M} = \bigwedge_{i \in N} \mathcal{P}_i$, the finest common coarsening of individuals' partitions $\mathcal{P}_i$, $i \in N$

Distributed Knowledge:

DE $= \{\omega \in \Omega : \bigcap_{i \in N} P_i(\omega) \subseteq E\}$

$=$ set of states in which everybody knows E

if they completely share their private information

Example

$$\Omega = \{1,2,3,4,5\} \quad \mathcal{P}_1 = \{\{1\},\{2,3\},\{4,5\}\} \quad \mathcal{P}_2 = \{\{1\},\{2\},\{3,4\},\{5\}\}$$

$$E = \{3,4,5\}$$

$K_1 E = \{4,5\}$, $K_2 E = \{3,4,5\} \Rightarrow KE = \{4,5\}$:

E is mutually known in $\omega = 4$ and 5

12/ $K_1 KE = \{4,5\}$, $K_2 KE = \{5\} \Rightarrow KKE = \{5\}$:

E is mutually known at order 2 in $\omega = 5$

$K_1 KKE = \emptyset$, $K_2 KKE = \{5\} \Rightarrow KKKE = \emptyset$:

E is never mutually known at order 3

$\Rightarrow E$ is never commonly known

On the contrary, $F = \{2,3,4,5\}$ is commonly known whenever F is realized

$$\mathcal{M} = \{\{1\},\{2,3,4,5\}\}$$

Beliefs and Consensus

Common prior probability distribution: $p \in \Delta(\Omega)$

Posterior belief of player i about $E \subseteq \Omega$ at $\omega \in \Omega$:

$$p(E \mid P_i(\omega)) = \frac{p(E \cap P_i(\omega))}{p(P_i(\omega))}$$

13/ ➥ *Differences in beliefs* between individuals only come from *asymmetric information*

In particular, individuals cannot agree to disagree: if their beliefs about an event E are commonly known, then these beliefs about E should be the same

Theorem. (We can't agree to disagree. Aumann, 1976) *Let N be a set of agents with the same prior beliefs on Ω with partitional (and correct) information about Ω. Let $E \subseteq \Omega$ be an event. If it is commonly known in some state $\omega \in \Omega$ that agent i's posterior belief about E is equal to q_i, for every $i \in N$, then these posterior beliefs are equal: $q_i = q_j$, for every $i, j \in N$*

Proof. Consider an agent $i \in N$ and the event "i's posterior belief about E is
14/ equal to q_i":

$$F_i = \{\omega \in \Omega : \Pr[E \mid P_i(\omega)] = q_i\}$$

F_i is commonly known at ω iff $M(\omega) \subseteq F_i$, i.e., $\Pr[E \mid P_i(\omega')] = q_i$ for every $\omega' \in M(\omega)$. Hence:

$$\Pr[E \mid M(\omega)] = q_i$$

because $M(\omega)$ is the union of disjoint cells $P_i(\omega')$ of $\mathcal{P}_i$ □

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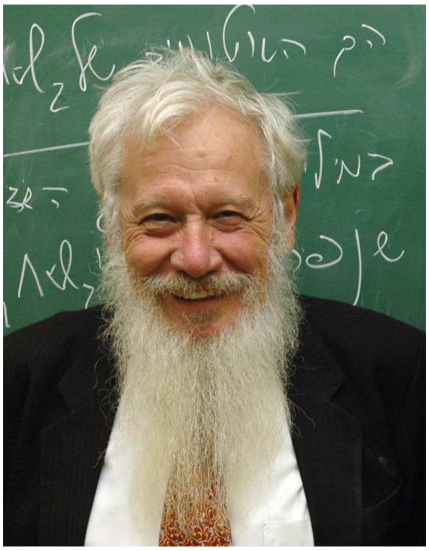

Figure 1: Robert Aumann (1930–), Nobel price in economics in 2005

✍ Show with a simple example that it can be commonly known between two individuals that they do not have the same posterior beliefs about some event E

✍ Show as in the proof before that it cannot be commonly known between two individuals that the posterior belief of the first individual about an event E is strictly larger than the posterior belief of the second individual

16/

✍ Show that the result is not valid if we replace "commonly known" by "mutually known" (take $\Omega = 1234$, p uniform, $\mathcal{P}_1 = \{12, 34\}$, $\mathcal{P}_2 = \{123, 4\}$, $E = 14$ and $\omega = 1$)

The result can easily be generalized from posterior beliefs to any rule (function) $f : 2^\Omega \to D$ which is *union-consistent*, i.e., such that for every disjoint events $E \subseteq \Omega$ and $F \subseteq \Omega$ (i.e., $E \cap F = \emptyset$), if $f(E) = f(F)$, then $f(E \cup F) = f(E) = f(F)$

17/ Examples: posterior beliefs, conditional expectation, decision maximizing an expected utility, . . .

If agents (publicly) communicate the values of such a function at their information sets, these values will become commonly known, and thus equal (**consensus**)

➥ "**We can't disagree forever**" (Geanakoplos and Polemarchakis, 1982; Cave, 1983)

✍ Show that the consensus is not necessarily the same if agents directly communicate their information (take $\Omega = 1234$, p uniform, $\mathcal{P}_1 = \{12, 34\}$, $\mathcal{P}_2 = \{13, 24\}$, $E = 14$, $f(\cdot) = \Pr(E \mid \cdot)$, and $\omega = 1$)

➥ If two detectives with the same preferences share the name of the suspect they would like to arrest, then after some time they will agree (reach a consensus), but not necessarily on the same suspect they would have arrested if they had shared all their clues (information)

18/

Bayesian Game

$$G = \langle N,\, \Omega,\, p,\, (\mathcal{P}_i)_i,\, (A_i)_i,\, (u_i)_i \rangle$$

- $N = \{1, \ldots, n\}$: set of *players*
- Ω: set of *states of the world*
- $p \in \Delta(\Omega)$: strictly positive *common prior* probability distribution
- $\mathcal{P}_i$: *information partition* of player i ($i = 1, \ldots, n$)
- A_i: nonempty set of *actions* of player i ($i = 1, \ldots, n$)
- $u_i : A_1 \times \cdots \times A_n \times \Omega \to \mathbb{R}$: *utility function* of player i ($i = 1, \ldots, n$)

Alternative equivalent representation (Harsanyi, 1967–1968):

Ω	➠	$T = T_1 \times \cdots \times T_n$: *type space*
$p \in \Delta(\Omega)$	➠	$p \in \Delta(T)$
$\mathcal{P}_i$	➠	T_i: type space of player i
$u_i(a; \omega)$	➠	$u_i(a; (t_1, \ldots, t_n))$

Particular Cases

Decision Problem

$$\langle \Omega, p, \mathcal{P}, A, u \rangle$$

Strategy (decision rule) $s : \Omega \to A$, measurable w.r.t. to $\mathcal{P}$

Proposition. In this model, a decision rule s is *ex-ante optimal*, i.e., s is a solution
21/ of

$$\max_{s} \sum_{\omega \in \Omega} p(\omega)\ u(s(\omega); \omega)$$

iff s is *interim optimal*, i.e., for every $\omega \in \Omega$, $s(\omega)$ is a solution of

$$\max_{s(\omega)} \sum_{\omega' \in \Omega} p(\omega' \mid P(\omega))\ u(s(\omega); \omega')$$

Proposition. In an individual decision problem, the value of information is always positive

Proof. If $\mathcal{P}$ is finer than $\mathcal{P}'$ then the set of strategies of the agent with $\mathcal{P}$ contains his set of strategies with $\mathcal{P}'$: $S' \subseteq S$. Hence:

$$\max_{s \in S} \mathrm{E}[u(s(\omega); \omega)] \geq \max_{s \in S'} \mathrm{E}[u(s(\omega); \omega)]$$

22/

➥ $\boxed{\text{more information}} \sim \boxed{\text{more strategies}}$

More generally, using the $\max\min$ property of Nash equilibria in zero-sum games, it can be shown that the value of information is always positive in these games

Bounded Rationality: we relax, for example, the negative introspection axiom

➥ The two previous propositions do not apply anymore

Example. $\Omega = \{1,2,3\}$, $P(1) = \{1,2\}$, $P(2) = \{2\}$, $P(3) = \{2,3\}$ $\Rightarrow$ negative introspection not verified anymore because $K\neg K\{2\} = K\neg\{2\} = K\{1,3\} = \emptyset$

In the following decision problem

23/

	Bet	Don't bet	Pr
ω_1	-2	0	$1/3$
ω_2	3	0	$1/3$
ω_3	-2	0	$1/3$

the interim optimal decision is BBB while the ex-ante optimal decision is DBD

In addition, the value of information is negative with the interim optimal decision rule (the payoff without information would be zero)

Perfect Information

$$P_i(\omega) = \{\omega\}, \quad \forall\, \omega \in \Omega$$

Symmetric Information

$$\mathcal{P}_i = \mathcal{P}_j, \quad \forall\, i,\, j \in N$$

24/

Independent Types

$$p\left[\bigcap_{i\in N} P_i(\omega)\right] = \prod_{i\in N} p\left[P_i(\omega)\right]$$

$$\Rightarrow \quad p((t_i)_{i\in N}) = p(t_1) \times \cdots \times p(t_n)$$

(Bayesian) Nash Equilibrium

- Pure strategy of player i:

$$s_i : \Omega \to A_i, \text{ measurable wrt } \mathcal{P}_i$$

- Mixed strategy of player i:

$$\sigma_i : \Omega \to \Delta(A_i), \text{ measurable wrt } \mathcal{P}_i$$

25/ ➢ *Pooling* strategy:

$$\sigma_i(\omega) = \sigma_i(\omega') \quad \forall\, \omega,\, \omega' \in \Omega$$

➢ *Separating* strategy:

$$s_i(\omega) \neq s_i(\omega') \quad \forall\, \omega,\, \omega' \text{ s.t. } \quad P_i(\omega) \neq P_i(\omega')$$

Set of pure (mixed) strategies of player i in G: S_i (Σ_i)

Definition. A **(Bayes) Nash Equilibrium** of the Bayesian game G is a Nash equilibrium of the normal form game

$$\widetilde{G} = \langle N,\, (\Sigma_i)_i,\, (\tilde{u}_i)_i \rangle$$

where $\tilde{u}_i(\sigma) \equiv \mathrm{E}[u_i(\sigma(\cdot);\cdot)] = \sum_{\omega\in\Omega} p(\omega) u_i(\sigma(\omega);\omega)$

i.e., a strategy profile $\sigma^* = (\sigma_i^*)_{i\in N}$ s.t.

$$\mathrm{E}[u_i(\sigma_i^*(\cdot), \sigma_{-i}^*(\cdot);\cdot)] \geq \mathrm{E}[u_i(\sigma_i(\cdot), \sigma_{-i}^*(\cdot);\cdot)]$$

26/ $\forall\, \sigma_i \in \Sigma_i,\ \forall\, i \in N$

$$\Leftrightarrow \quad \sum_{\omega'\in\Omega} p(\omega' \mid P_i(\omega)) u_i(\sigma_i^*(\omega), \sigma_{-i}^*(\omega');\omega') \geq \sum_{\omega'\in\Omega} p(\omega' \mid P_i(\omega)) u_i(a_i, \sigma_{-i}^*(\omega');\omega')$$

$\forall\, a_i \in A_i,\ \forall\, \omega \in \Omega,\ \forall\, i \in N$

In a game, the value of information may be negative.

$$\Omega = \{\omega_1, \omega_2\}, \quad p(\omega_1) = p(\omega_2) = 1/2$$

ω_1	a	b	ω_2	a	b
a	$(0,0)$	$(6,-3)$	a	$(-20,-20)$	$(-7,-16)$
b	$(-3,6)$	$(5,5)$	b	$(-16,-7)$	$(-5,-5)$

❶ The two players are uninformed: $\mathcal{P}_1 = \mathcal{P}_2 = \{\{\omega_1, \omega_2\}\}$

27/ $\Rightarrow$ Unique NE: $(b,b) \Rightarrow \mathbf{(0,0)}$

❷ The two players are informed: $\mathcal{P}_1 = \mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2\}\}$

$\Rightarrow$ Unique NE: $((a,a) \mid \omega_1)$, $((b,b) \mid \omega_2) \Rightarrow \mathbf{(-2.5, -2.5)}$

❸ Only player 1 is informed: $\mathcal{P}_1 = \{\{\omega_1\}, \{\omega_2\}\}$, $\mathcal{P}_2 = \{\{\omega_1, \omega_2\}\}$

$\Rightarrow$ Unique NE: $((a,a) \mid \omega_1)$, $((b,a) \mid \omega_2) \Rightarrow \mathbf{(-8, -3.5)}$

APPLICATIONS

28/

Not Trade / No Bet Theorem

Example. 2 players can bet on the realization of a state in $\Omega = \{\omega_1, \omega_2, \omega_3\}$, with a uniform prior probability distribution

$$\text{Payoffs:} \quad \begin{cases} \omega_1 \longrightarrow (2, -2) \\ \omega_2 \longrightarrow (-3, 3) \\ \omega_3 \longrightarrow (5, -5) \end{cases} \qquad \text{Information:} \quad \begin{cases} \mathcal{P}_1 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\} \\ \mathcal{P}_2 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\} \end{cases}$$

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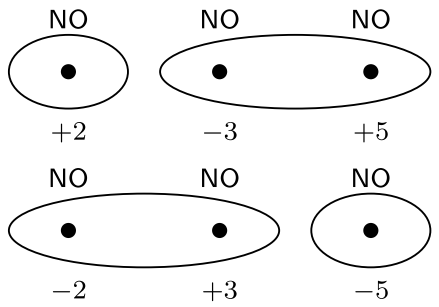

$\Rightarrow$ Unique NE: no bet

General Case

A *zero-sum bet* $x : \Omega \to \mathbb{R}$ is proposed to the players

They decide simultaneously to bet (action B) or not to bet (action D)

Payoffs: $(0, 0)$

Payoffs at ω if both players bet: $(x(\omega),\ -x(\omega))$

30/

No Bet Theorem. Whatever the (correct and partitional) information structure, no player, whatever his information set, can expect strictly positive payoffs at a Nash equilibrium

$\Rightarrow$ Pure speculation cannot be explained by asymmetric information only

Important assumptions:

- Every player is rational at **every** state of the world
  (⇒ common knowledge of rationality)
  Previous example: if player 2 is not rational at ω_3 then all players may bet in every state

31/ ⇒ at ω_1 everybody bets and everybody knows that everybody is rational (but rationality is not commonly known)

- **Common** prior probability distribution
  (differences in beliefs only come from asymmetric information)

- **Partitional** information structure

  For example, in the following situation

	Bet	Don't Bet	$\Pr$
ω_1	-2	0	$1/3$
ω_2	3	0	$1/3$
ω_3	-2	0	$1/3$

32/

with $P_1(1) = \{1, 2\}$, $P_1(2) = \{2\}$, $P_1(3) = \{2, 3\}$ and $\mathcal{P}_2 = \{\Omega\}$, players bet in every state

Reinterpretation of Mixed Strategies

Harsanyi (1973): the mixed strategy of player i represents others' uncertainty about the action chosen by player i. This uncertainty is due to the fact that player i has a small private information about his preference

Example.

	a	b
a	$3+t_1,\, 3+t_2$	$3+t_1,\, 0$
b	$0,\, 3+t_2$	$4,\, 4$

33/

☞ NE if $t_1 = t_2 = 0$: (a,a), (b,b) and $\sigma_1(a) = \sigma_2(a) = 1/4$

☞ Incomplete information: t_1, t_2 i.i.d. $\mathcal{U}[0,T]$

Consider the following (symmetric) pure strategies:

$$\begin{array}{ll} \text{Play } a & \text{if } t_i > t^* \\ \text{Play } b & \text{if } t_i \leq t^* \end{array}$$

	a	b
a	$3+t_1,\, 3+t_2$	$3+t_1,\, 0$
b	$0,\, 3+t_2$	$4,\, 4$

Belief of each player about the other player's action:

$$\mu(a) = \frac{T - t^*}{T} \qquad\qquad \mu(b) = \frac{t^*}{T}$$

34/ $\Rightarrow$ Expected payoff of player i as a function of his action:

$$a \stackrel{i}{\longrightarrow} 3 + t_i \qquad\qquad b \stackrel{i}{\longrightarrow} 4\,t^*/T$$

so $a \succ_i b \quad \Leftrightarrow \quad 3 + t_i > \frac{4\,t^*}{T} \quad \Leftrightarrow \quad t_i > \frac{4\,t^* - 3T}{T}$

The original strategy is a NE of the Bayesian game if $\frac{4\,t^*-3T}{T} = t^*$, i.e., $t^* = \frac{3T}{4-T}$, so

$$\mu(a) = \frac{T - t^*}{T} = 1 - \frac{3}{4-T} \stackrel{(T \to 0)}{\longrightarrow} \; 1/4 = \sigma_i(a)$$

Harsanyi (1973) shows, more generally, that every Nash equilibrium (especially in mixed strategies) of a normal form game can "almost always" be obtained as the limit of a pure strategy NE of such a perturbed game with incomplete information when the prior uncertainty (T) tends to 0

35/ ➥ Stability of mixed strategies

Correlation and communication

Possible interpretation of mixed strategy equilibria: players' actions depend on *independent private signals* (mood, position of the second hand of their watch, ...) that do not affect players' payoffs

Example: Battle of sexes.

36/

	a	b
a	$(3, 2)$	$(1, 1)$
b	$(0, 0)$	$(2, 3)$

The mixed strategy NE, $((3/4, 1/4), (1/4, 3/4))$, generates the same outcome (so, the same payoffs $(3/2, 3/2)$) as a pure strategy NE of the Bayesian game in which each player has two possible types, t_i^a, t_i^b, that are independent and payoff irrelevant, where $\Pr(t_1^a) = \Pr(t_2^b) = 3/4$, $\Pr(t_1^b) = \Pr(t_2^a) = 1/4$, $\sigma_i(t_i^a) = a$, and $\sigma_i(t_i^b) = b$

✍ Write the previous information structure with information partitions

What happens if players can observe *correlated signals*, or simply *common* (*public*) signals?

Example: public observation of a coin flip ($\mathcal{P}_1 = \mathcal{P}_2 = \{H, T\}$)

➡ New equilibrium in the battle of sexes game, e.g., (a, a) if H and (b, b) if T

37/ ☞ **Public Correlated Equilibrium**

The induced distribution of actions $\mu = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, and the payoffs $(5/2, 5/2)$

cannot be obtained as a Nash equilibrium of the original game

We can also have an intermediate situation between independent signals (NE in mixed strategies) and public signals (public correlated equilibrium = convex combination of NE)

For example, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $p(\omega) = 1/3$, and

38/

$$\mathcal{P}_1 = \{\underbrace{\{\omega_1, \omega_2\}}_{a}, \underbrace{\{\omega_3\}}_{b}\}$$
$$\mathcal{P}_2 = \{\underbrace{\{\omega_1\}}_{a}, \underbrace{\{\omega_2, \omega_3\}}_{b}\}$$

generates the distribution $\mu = \begin{pmatrix} 1/3 & 1/3 \\ 0 & 1/3 \end{pmatrix}$, and the payoffs $(2, 2)$

Definition. (Aumann, 1974) A **correlated equilibrium** (CE) of the normal form game

$$\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$$

is a pure strategy NE of the Bayesian game

$$\langle N, \Omega, p, (\mathcal{P}_i)_i, (A_i)_i, (u_i)_i\rangle$$

where players' payoffs do not depend on the state of the world ($u_i(a;\omega) = u_i(a)$),
39/ i.e., a profile of pure strategies $s = (s_1, \dots, s_n)$ such that, for every player $i \in N$ and strategy r_i of player i:

$$\sum_{\omega\in\Omega} p(\omega)\ u_i(s_i(\omega), s_{-i}(\omega)) \geq \sum_{\omega\in\Omega} p(\omega)\ u_i(r_i(\omega), s_{-i}(\omega))$$

➥ *Correlated equilibrium outcome* or distribution $\mu \in \Delta(A)$, where $\mu(a) = p(\{\omega \in \Omega : s(\omega) = a\})$

➥ *Correlated equilibrium payoff* $\sum_{a\in A} \mu(a)u_i(a),\ i = 1, \dots, n$

In the battle of sexes game, every correlated equilibrium payoff we have seen belongs to the convex hull of the set of NE payoffs:

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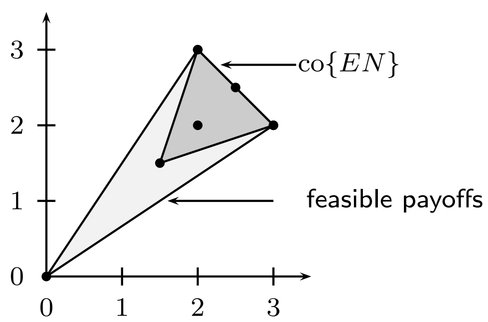

☞ **But the set of CE payoffs does not always belong to the convex hull of the set of NE payoffs**

$$\mathcal{P}_1 = \{\underbrace{\{\omega_1, \omega_2\}}_{a}, \underbrace{\{\omega_3\}}_{b}\}$$

$$\mathcal{P}_2 = \{\underbrace{\{\omega_1\}}_{a}, \underbrace{\{\omega_2, \omega_3\}}_{b}\}$$

	a	b
a	$(2,7)$	$(6,6)$
b	$(0,0)$	$(7,2)$

Chicken Game

➡ Correlated equilibrium payoffs $(5,5) \notin \text{co}\{\text{EN}\}$

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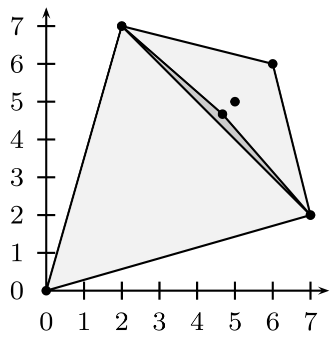

A CE may even Pareto dominate all NE

For example, in the game

0,0	1,2	2,1
2,1	0,0	1,2
1,2	2,1	0,0

42/

the unique NE distribution is $\begin{pmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{pmatrix}$, with the expected payoff $\frac{1+2}{3} = 1$ for each player, while the CE distribution $\begin{pmatrix} 0 & 1/6 & 1/6 \\ 1/6 & 0 & 1/6 \\ 1/6 & 1/6 & 0 \end{pmatrix}$ gives the expected payoff $3/2$ for each player

Proposition.

① *In the definition of a CE we can allow for mixed strategies in the Bayesian game, this does not enlarge the set of CE outcomes. In particular, a mixed strategy NE outcome is a CE outcome*

② *Every convex combination of CE outcomes is a CE outcome*

Proof. It suffices to construct the appropriate information system (see also Osborne and Rubinstein, 1994, propositions 45.3 and 46.2) □

43/

Information systems used in the previous examples:

➢ Set of states $\Omega \subseteq$ set of action profiles A

➢ Each player is only informed about his action

➥ **Canonical Information System**

Proposition. *Every correlated equilibrium outcome of a normal form game* $\langle N, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ *is a* **canonical correlated equilibrium** *outcome, where the information structure and strategies are given by:*

- $\Omega = A$
- $\mathcal{P}_i = \{\{a \in A : a_i = b_i\} : b_i \in A_i\}$ *for every* $i \in N$
- $s_i(a) = a_i$ *for every* $a \in A$ *and* $i \in N$

44/

➥ **"Revelation principle" for complete information games:**

Other possible interpretation: Every correlated equilibrium outcome can be achieved with a mediator who makes private recommendations to the players, and no player has an incentive to deviate from the mediator's recommendation

Set of correlated equilibrium outcomes $\mu = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ of the game

	a	b
a	$(2,7)$	$(6,6)$
b	$(0,0)$	$(7,2)$

45/ Incentive constraints:

$$\text{Player 1} \quad \begin{cases} 2\mu_1 + 6\mu_2 \geq 7\mu_2 \\ 7\mu_4 \geq 2\mu_3 + 6\mu_4 \end{cases} \qquad \text{Player 2} \quad \begin{cases} 7\mu_1 \geq 6\mu_1 + 2\mu_3 \\ 6\mu_2 + 2\mu_4 \geq 7\mu_2 \end{cases}$$

$$\iff \quad \begin{cases} \mu_2 \leq 2\mu_1 \\ \mu_2 \leq 2\mu_4 \end{cases} \quad \text{and} \quad \begin{cases} 2\mu_3 \leq \mu_4 \\ 2\mu_3 \leq \mu_1 \end{cases}$$

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