Incomplete Information and Bayesian Games

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(October 27, 2008)

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- Information structure, knowledge and common knowledge, beliefs

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  - Correlation and communication

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☞ Contractual relationships: The principal (insurer, employer, regulator, . . .) does not know the "type" of the agent(s)

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Information set of player i at ω: $P_i(\omega) =$ element of $\mathcal{P}_i$ containing ω

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$$P_i(00) \quad = \quad \ldots \quad = \quad P_i(09) \quad = \quad \{00, 01, \ldots, 09\}$$

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Correct ($\omega \in P_i(\omega)$ for every $\omega \in \Omega$)

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$\Omega = \{B, M\}$ and the agent only remembers good news:

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$B \in P_i(M)$ but $P_i(B) \neq P_i(M)$ (imperfect introspection)

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☞ No player is more informed than the other

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$\neg K_iE \subseteq K_i\neg K_iE$ (negative introspection axiom): if an agent does not know E, then he knows that he does not know E (most restrictive axiom)

Example.

Example. $\Omega = \{1, 2, 3, 4\}$

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$E = \{1, 3\} \Rightarrow K_1 E = \{1\}$,

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$E = \{1,3\} \Rightarrow K_1 E = \{1\}$, $K_2 E = \emptyset$, $K_1 \neg E = \{2\}$

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Interactive Knowledge

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Mutual/shared Knowledge:

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KE

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where $M(\omega)$ is the cell of the common knowledge partition ("*Meet*"),
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Distributed Knowledge:

$$\begin{aligned} DE &= \{\omega \in \Omega : \textstyle\bigcap_{i \in N} P_i(\omega) \subseteq E\} \\ &= \text{set of states in which everybody knows } E \\ &\quad \text{if they completely share their private information} \end{aligned}$$

Example

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$$\Omega = \{1, 2, 3, 4, 5\}$$

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E is mutually known at order 2 in $\omega = 5$

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$K_1 E = \{4, 5\}$, $K_2 E = \{3, 4, 5\} \Rightarrow KE = \{4, 5\}$:

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$$E = \{3, 4, 5\}$$

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$K_1 KE = \{4, 5\}$, $K_2 KE = \{5\} \Rightarrow KKE = \{5\}$:

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$K_1 KKE = \emptyset$, $K_2 KKE = \{5\} \Rightarrow KKKE = \emptyset$:

E is never mutually known at order 3

$\Rightarrow E$ is never commonly known

On the contrary, $F = \{2, 3, 4, 5\}$ is commonly known whenever F is realized

$$\mathcal{M} = \{\{1\}, \{2, 3, 4, 5\}\}$$

Beliefs and Consensus

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Common prior probability distribution: $p \in \Delta(\Omega)$

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➥ Differences in beliefs between individuals only come from asymmetric information

In particular, individuals cannot agree to disagree: if their beliefs about an event E are commonly known, then these beliefs about E should be the same

Theorem. (We can't agree to disagree. Aumann, 1976) *Let N be a set of agents with the same prior beliefs on Ω with partitional (and correct) information about Ω. Let $E \subseteq \Omega$ be an event. If it is commonly known in some state $\omega \in \Omega$ that agent i's posterior belief about E is equal to q_i, for every $i \in N$, then these posterior beliefs are equal: $q_i = q_j$, for every $i, j \in N$*

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Proof. Consider an agent $i \in N$ and the event "i's posterior belief about E is equal to q_i":

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Proof. Consider an agent $i \in N$ and the event "i's posterior belief about E is equal to q_i":

$$F_i = \{\omega \in \Omega : \Pr[E \mid P_i(\omega)] = q_i\}$$

F_i is commonly known at ω iff $M(\omega) \subseteq F_i$, i.e., $\Pr[E \mid P_i(\omega')] = q_i$ for every $\omega' \in M(\omega)$. Hence:

$$\Pr[E \mid M(\omega)] = q_i$$

because $M(\omega)$ is the union of disjoint cells $P_i(\omega')$ of $\mathcal{P}_i$ □

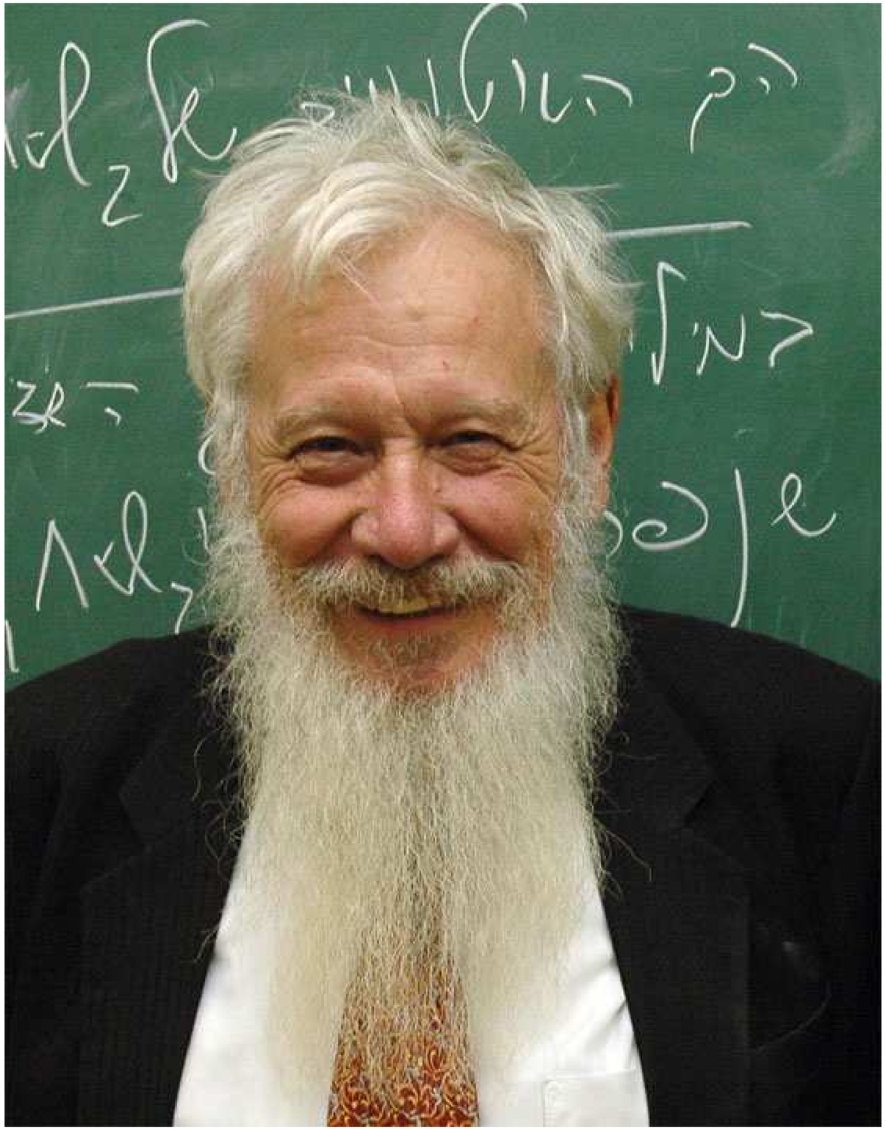

Figure 1: Robert Aumann (1930–), Nobel price in economics in 2005

✍ Show with a simple example that it can be commonly known between two individuals that they do not have the same posterior beliefs about some event E

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✍ Show as in the proof before that it cannot be commonly known between two individuals that the posterior belief of the first individual about an event E is strictly larger than the posterior belief of the second individual

✍ Show that the result is not valid if we replace "commonly known" by "mutually known" (take $\Omega = 1234$, p uniform, $\mathcal{P}_1 = \{12, 34\}$, $\mathcal{P}_2 = \{123, 4\}$, $E = 14$ and $\omega = 1$)

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The result can easily be generalized from posterior beliefs to any rule (function) $f : 2^\Omega \to D$ which is union-consistent, i.e., such that for every disjoint events $E \subseteq \Omega$ and $F \subseteq \Omega$ (i.e., $E \cap F = \emptyset$), if $f(E) = f(F)$, then $f(E \cup F) = f(E) = f(F)$

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Examples: posterior beliefs, conditional expectation, decision maximizing an expected utility, ...

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If agents (publicly) communicate the values of such a function at their information sets, these values will become commonly known, and thus equal (**consensus**)

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Examples: posterior beliefs, conditional expectation, decision maximizing an expected utility, . . .

If agents (publicly) communicate the values of such a function at their information sets, these values will become commonly known, and thus equal (**consensus**)

➥ "**We can't disagree forever**" (Geanakoplos and Polemarchakis, 1982; Cave, 1983)

✍ Show that the consensus is not necessarily the same if agents directly communicate their information (take $\Omega = 1234$, p uniform, $\mathcal{P}_1 = \{12, 34\}$, $\mathcal{P}_2 = \{13, 24\}$, $E = 14$, $f(\cdot) = \Pr(E \mid \cdot)$, and $\omega = 1$)

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➥ If two detectives with the same preferences share the name of the suspect they would like to arrest, then after some time they will agree (reach a consensus), but not necessarily on the same suspect they would have arrested if they had shared all their clues (information)

Bayesian Game

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$$G = \langle N,\, \Omega,\, p,\, (\mathcal{P}_i)_i,\, (A_i)_i,\, (u_i)_i \rangle$$

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- A_i: nonempty set of actions of player i $(i = 1, \ldots, n)$
- $u_i : A_1 \times \cdots \times A_n \times \Omega \to \mathbb{R}$: utility function of player i $(i = 1, \ldots, n)$

Alternative equivalent representation (Harsanyi, 1967–1968):

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Ω	➠	$T = T_1 \times \cdots \times T_n$: type space
$p \in \Delta(\Omega)$	➠	$p \in \Delta(T)$
$\mathcal{P}_i$	➠	T_i: type space of player i
$u_i(a;\omega)$	➠	$u_i(a;(t_1,\ldots,t_n))$

Particular Cases

Decision Problem

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Proposition. In this model, a decision rule s is ex-ante optimal, i.e., s is a solution of

$$\max_s \sum_{\omega \in \Omega} p(\omega)\ u(s(\omega); \omega)$$

iff s is interim optimal, i.e., for every $\omega \in \Omega$, $s(\omega)$ is a solution of

$$\max_{s(\omega)} \sum_{\omega' \in \Omega} p(\omega' \mid P(\omega))\ u(s(\omega); \omega')$$

Proposition. In an individual decision problem, the value of information is always positive

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Proof. If $\mathcal{P}$ is finer than $\mathcal{P}'$ then the set of strategies of the agent with $\mathcal{P}$ contains his set of strategies with $\mathcal{P}'$: $S' \subseteq S$. Hence:

$$\max_{s \in S} \mathrm{E}[u(s(\omega); \omega)] \geq \max_{s \in S'} \mathrm{E}[u(s(\omega); \omega)]$$

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More generally, using the $\max\min$ property of Nash equilibria in zero-sum games, it can be shown that the value of information is always positive in these games

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In addition, the value of information is negative with the interim optimal decision rule (the payoff without information would be zero)

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Independent Types

$$p\left[\bigcap_{i \in N} P_i(\omega)\right] = \prod_{i \in N} p\,[P_i(\omega)]$$

➠ $p((t_i)_{i \in N}) = p(t_1) \times \cdots \times p(t_n)$

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Set of pure (mixed) strategies of player i in G: S_i (Σ_i)

Definition. A **(Bayes) Nash Equilibrium** of the Bayesian game G is a Nash equilibrium of the normal form game

$$\widetilde{G} = \langle N,\, (\Sigma_i)_i,\, (\tilde{u}_i)_i \rangle$$

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$$\Leftrightarrow \quad \sum_{\omega' \in \Omega} p(\omega' \mid P_i(\omega)) u_i(\sigma_i^*(\omega), \sigma_{-i}^*(\omega');\omega') \geq \sum_{\omega' \in \Omega} p(\omega' \mid P_i(\omega)) u_i(a_i, \sigma_{-i}^*(\omega');\omega')$$

$\forall\ a_i \in A_i,\ \forall\ \omega \in \Omega,\ \forall\ i \in N$

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a	$(-10,-10)$	$(-0.5,-9.5)$
b	$(-9.5,-0.5)$	$\mathbf{(0,0)}$

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APPLICATIONS

Not Trade / No Bet Theorem

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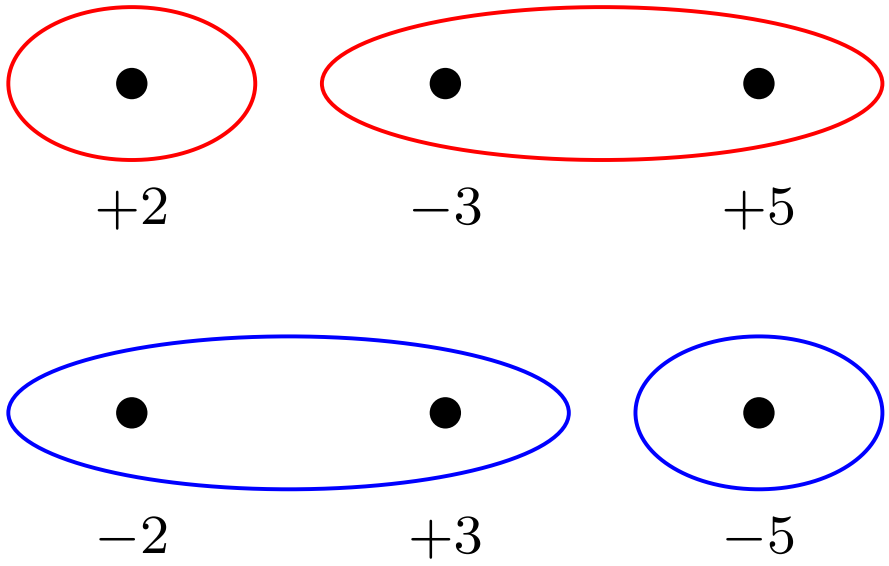

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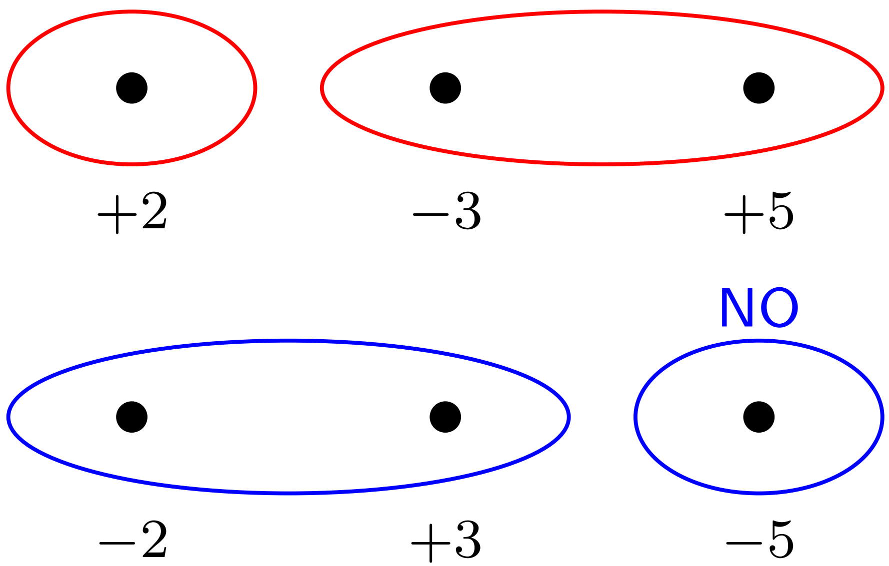

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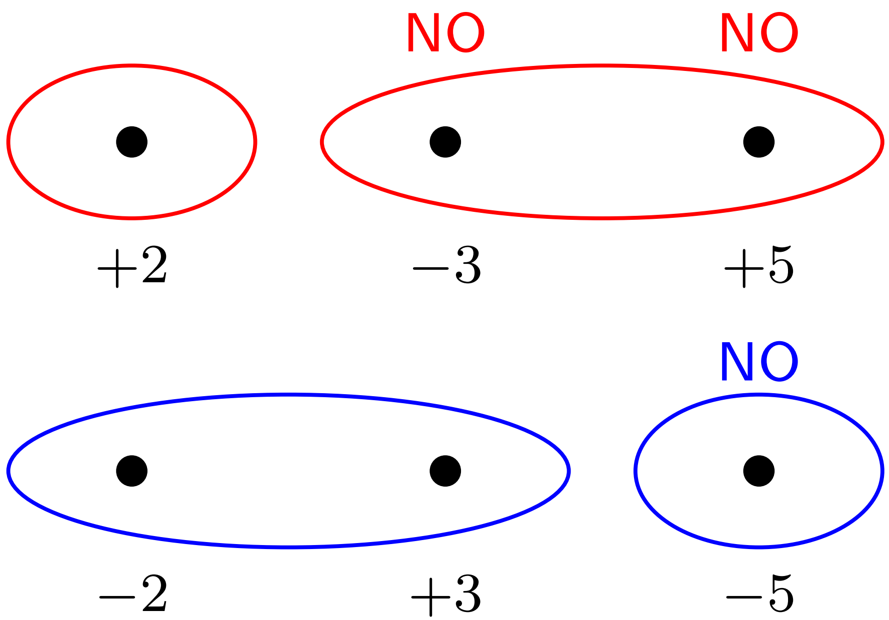

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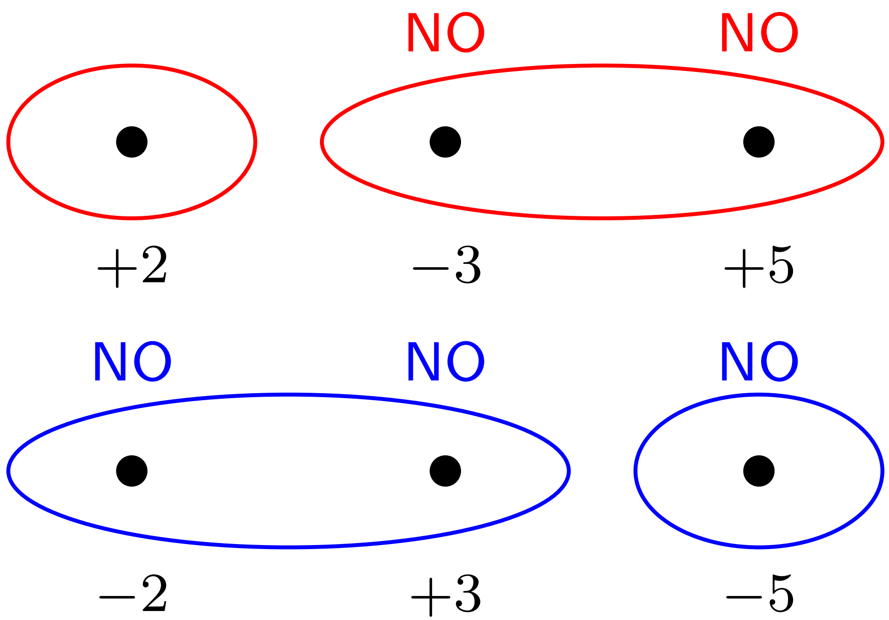

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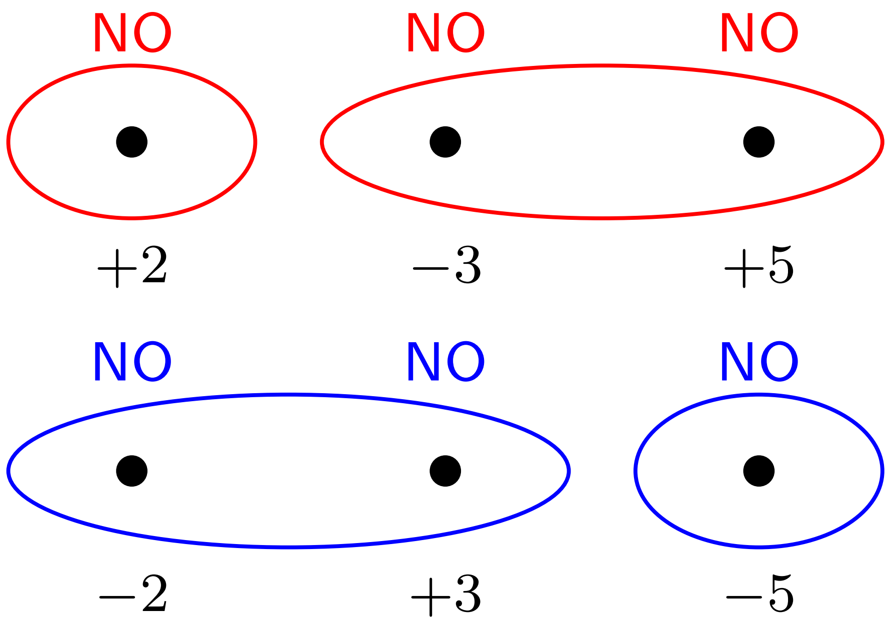

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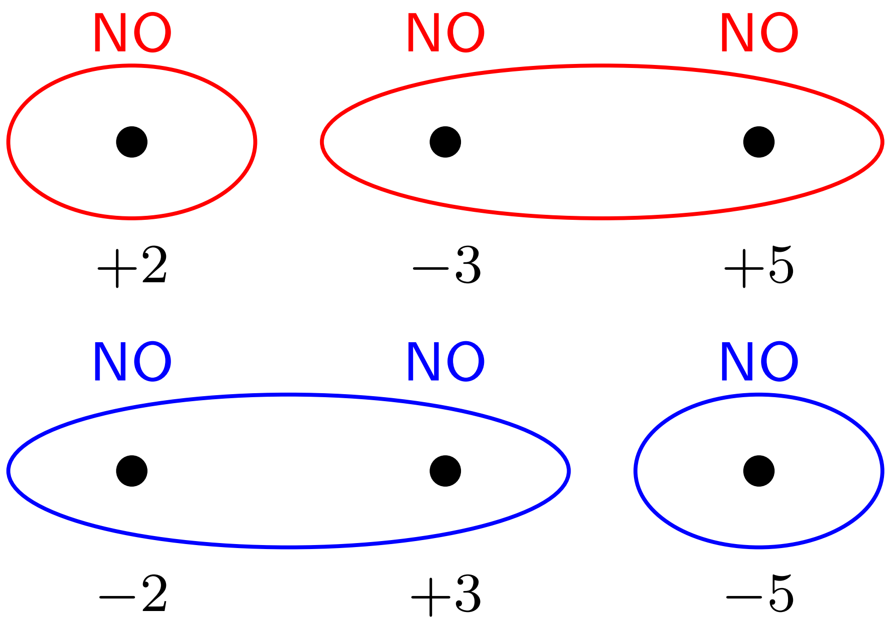

$\Rightarrow$ Unique NE: no bet

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$\Rightarrow$ Pure speculation cannot be explained by asymmetric information only

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- **Common** prior probability distribution

  (differences in beliefs only come from asymmetric information)

- **Partitional** information structure

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ω_2	3	0	$1/3$
ω_3	-2	0	$1/3$

  with $P_1(1) = \{1, 2\}$, $P_1(2) = \{2\}$, $P_1(3) = \{2, 3\}$ and $\mathcal{P}_2 = \{\Omega\}$, players bet in every state

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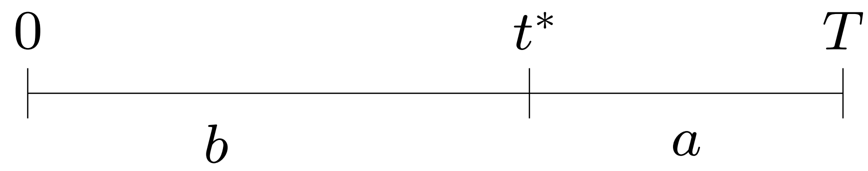
Consider the following (symmetric) pure strategies:

$$\text{Play } a \quad \text{if } t_i > t^*$$
$$\text{Play } b \quad \text{if } t_i \leq t^*$$

	a	b
a	$3+t_1,\ 3+t_2$	$3+t_1,\ 0$
b	$0,\ 3+t_2$	$4,\ 4$



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Belief of each player about the other player's action:

$$\mu(a) = \frac{T - t^*}{T} \qquad \mu(b) = \frac{t^*}{T}$$

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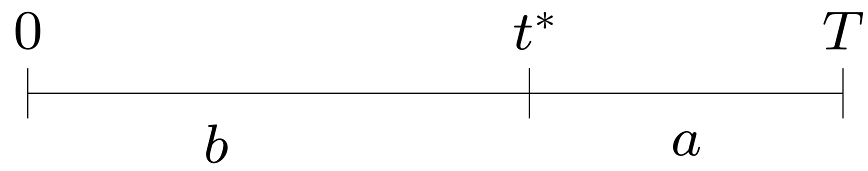

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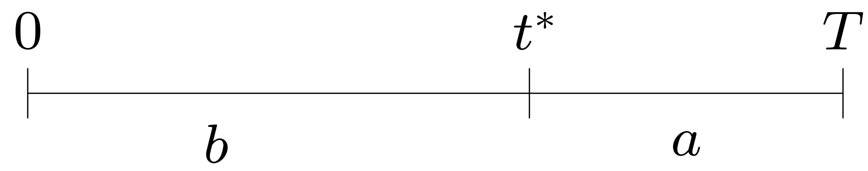

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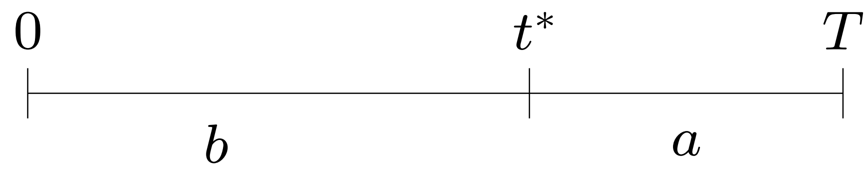

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➥ Stability of mixed strategies

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Possible interpretation of mixed strategy equilibria: players' actions depend on independent private signals (mood, position of the second hand of their watch, ...) that do not affect players' payoffs

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✍ Write the previous information structure with information partitions

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☞ **Public Correlated Equilibrium**

The induced distribution of actions $\mu = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, and the payoffs $(5/2, 5/2)$

cannot be obtained as a Nash equilibrium of the original game

We can also have an intermediate situation between independent signals (NE in mixed strategies) and public signals (public correlated equilibrium = convex combination of NE)

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For example, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $p(\omega) = 1/3$, and

$$\mathcal{P}_1 = \{\underbrace{\{\omega_1, \omega_2\}}_{a}, \underbrace{\{\omega_3\}}_{b}\}$$
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generates the distribution $\mu = \begin{pmatrix} 1/3 & 1/3 \\ 0 & 1/3 \end{pmatrix}$, and the payoffs $(2, 2)$

Definition. (Aumann, 1974) A **correlated equilibrium** (CE) of the normal form game

$$\langle N,\, (A_i)_{i\in N},\, (u_i)_{i\in N}\rangle$$

is a pure strategy NE of the Bayesian game

$$\langle N,\, \Omega,\, p,\, (\mathcal{P}_i)_i,\, (A_i)_i,\, (u_i)_i\rangle$$

where players' payoffs do not depend on the state of the world $(u_i(a;\omega) = u_i(a))$, i.e., a profile of pure strategies $s = (s_1, \ldots, s_n)$ such that, for every player $i \in N$ and strategy r_i of player i:

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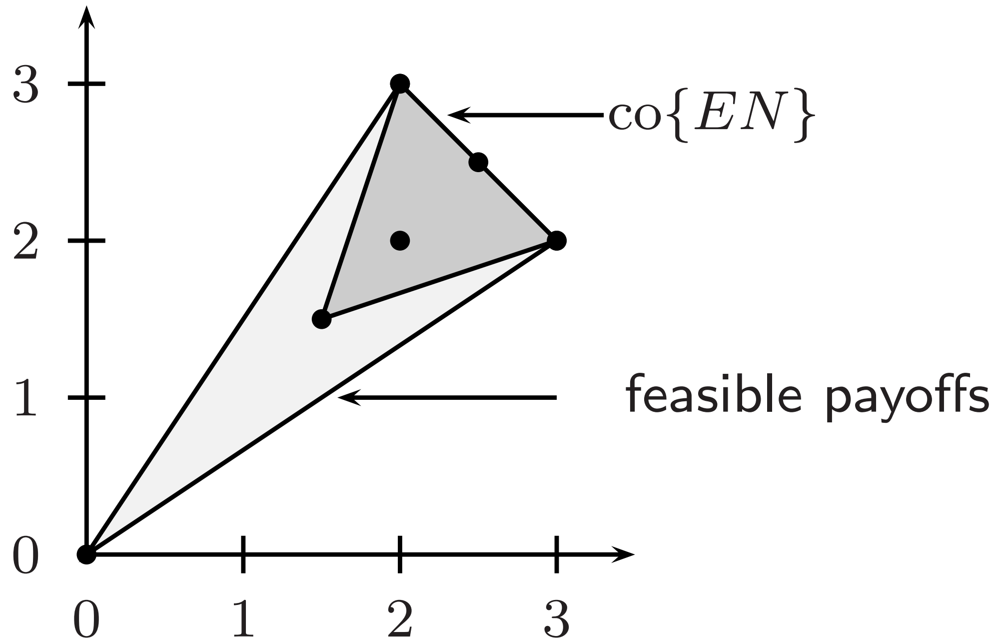
➥ Correlated equilibrium payoff $\sum_{a \in A} \mu(a) u_i(a)$, $i = 1, \ldots, n$

In the battle of sexes game, every correlated equilibrium payoff we have seen belongs to the convex hull of the set of NE payoffs:



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	a	b
a	$(2, 7)$	$(6, 6)$
b	$(0, 0)$	$(7, 2)$

Chicken Game

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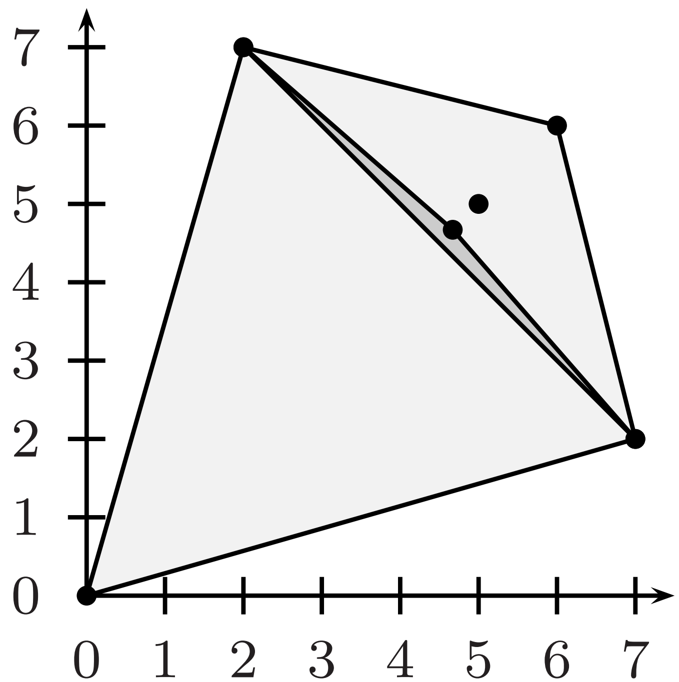
Chicken Game

➡ Correlated equilibrium payoffs $(5,5) \notin \mathrm{co}\{\mathsf{EN}\}$



A CE may even Pareto dominate all NE

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For example, in the game

0,0	1,2	2,1
2,1	0,0	1,2
1,2	2,1	0,0

the unique NE distribution is $\begin{pmatrix} 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \\ 1/9 & 1/9 & 1/9 \end{pmatrix}$, with the expected payoff $\frac{1+2}{3} = 1$ for each player, while the CE distribution $\begin{pmatrix} 0 & 1/6 & 1/6 \\ 1/6 & 0 & 1/6 \\ 1/6 & 1/6 & 0 \end{pmatrix}$ gives the expected payoff $3/2$ for each player

Proposition.

① *In the definition of a CE we can allow for mixed strategies in the Bayesian game, this does not enlarge the set of CE outcomes. In particular, a mixed strategy NE outcome is a CE outcome*

② *Every convex combination of CE outcomes is a CE outcome*

Proof. It suffices to construct the appropriate information system (see also Osborne and Rubinstein, 1994, propositions 45.3 and 46.2) □

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- Set of states $\Omega \subseteq$ set of action profiles A
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➥ **Canonical Information System**

Proposition. *Every correlated equilibrium outcome of a normal form game* $\langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ *is a* **canonical correlated equilibrium** *outcome, where the information structure and strategies are given by:*

- $\Omega = A$
- $\mathcal{P}_i = \{\{a \in A : a_i = b_i\} : b_i \in A_i\}$ *for every* $i \in N$
- $s_i(a) = a_i$ *for every* $a \in A$ *and* $i \in N$

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➥ **"Revelation principle" for complete information games:**

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- $\Omega = A$
- $\mathcal{P}_i = \{\{a \in A : a_i = b_i\} : b_i \in A_i\}$ *for every* $i \in N$
- $s_i(a) = a_i$ *for every* $a \in A$ *and* $i \in N$

➥ **"Revelation principle" for complete information games:**

Other possible interpretation: Every correlated equilibrium outcome can be achieved with a mediator who makes private recommendations to the players, and no player has an incentive to deviate from the mediator's recommendation

Set of correlated equilibrium outcomes $\mu = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ of the game

	a	b
a	$(2,7)$	$(6,6)$
b	$(0,0)$	$(7,2)$

Set of correlated equilibrium outcomes $\mu = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix}$ of the game

	a	b
a	$(2, 7)$	$(6, 6)$
b	$(0, 0)$	$(7, 2)$

Incentive constraints:

Player 1 $\begin{cases} 2\mu_1 + 6\mu_2 \geq 7\mu_2 \\ 7\mu_4 \geq 2\mu_3 + 6\mu_4 \end{cases}$ Player 2 $\begin{cases} 7\mu_1 \geq 6\mu_1 + 2\mu_3 \\ 6\mu_2 + 2\mu_4 \geq 7\mu_2 \end{cases}$

$$\Longleftrightarrow \quad \begin{cases} \mu_2 \leq 2\mu_1 \\ \mu_2 \leq 2\mu_4 \end{cases} \quad \text{and} \quad \begin{cases} 2\mu_3 \leq \mu_4 \\ 2\mu_3 \leq \mu_1 \end{cases}$$

References

AUMANN, R. J. (1974): "Subjectivity and Correlation in Randomized Strategies," Journal of Mathematical Economics, 1, 67–96.

——— (1976): "Agreeing to Disagree," The Annals of Statistics, 4, 1236–1239.

CAVE, J. A. K. (1983): "Learning to Agree," Economics Letters, 12, 147–152.

GEANAKOPLOS, J. AND H. M. POLEMARCHAKIS (1982): "We Can't Disagree Forever," Journal of Economic Theory, 28, 192–200.

HARSANYI, J. C. (1967–1968): "Games with Incomplete Information Played by Bayesian Players. Parts I, II, III," Management Science, 14, 159–182, 320–334, 486–502.

——— (1973): "Games with Randomly Disturbed Payoffs: A New Rationale for Mixed Strategy Equilibrium Points," International Journal of Game Theory, 2, 1–23.

LEWIS, D. (1969): Convention, a Philosophical Study, Harvard University Press, Cambridge, Mass.

OSBORNE, M. J. AND A. RUBINSTEIN (1994): A Course in Game Theory, Cambridge, Massachusetts: MIT Press.