Extensive Form Games

(Dynamic Games)

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Outline

(September 3, 2007)

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- Game tree, information and memory

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- Repeated Games (of complete information with perfect monitoring)

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- Game tree, information and memory
- Strategies and reduced games
- Subgame perfect equilibrium
- Repeated Games (of complete information with perfect monitoring)
- Negotiation: Strategic approach

Extensive form game: taking into account the detailed temporal structure of the decision problem (game tree), the evolution of information ("knowledge"), beliefs, and action sets ("ability")

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- Entry deterrence, reputation

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But we will see that every extensive form game can be written in normal form, by appropriately defining players' strategies

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- Probabilities of Nature's moves

Examples

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Prisoner Dilemma

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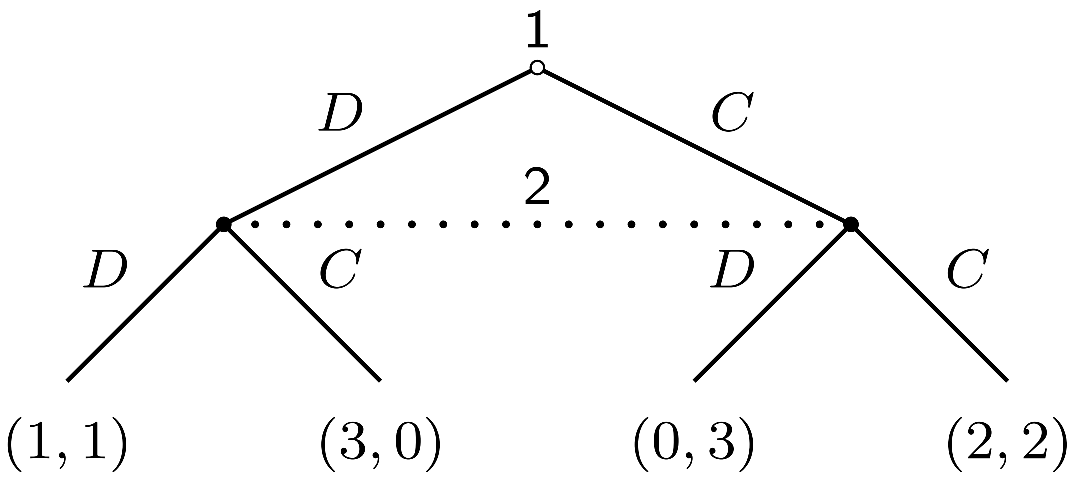

Examples

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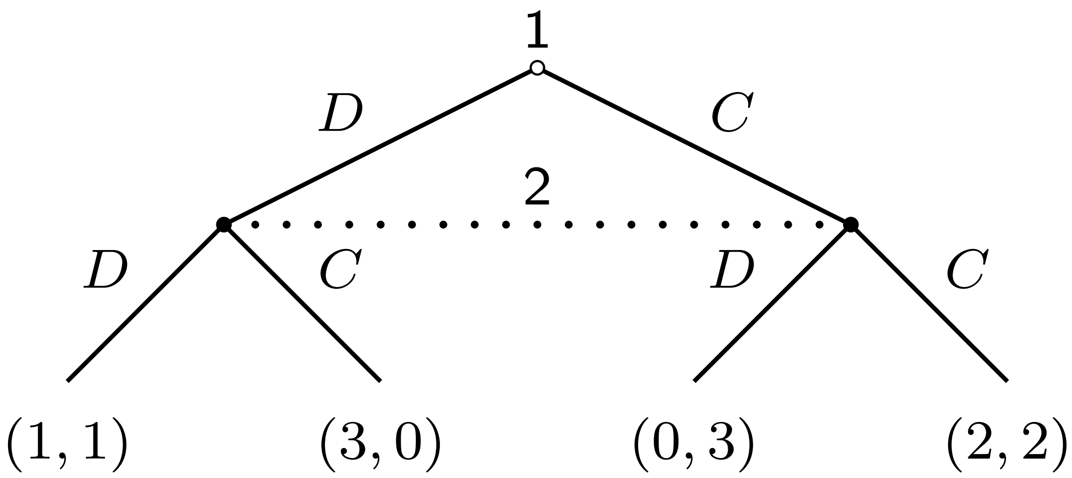

✍ Two repetitions with perfect monitoring ...

Ultimatum Game (finite)

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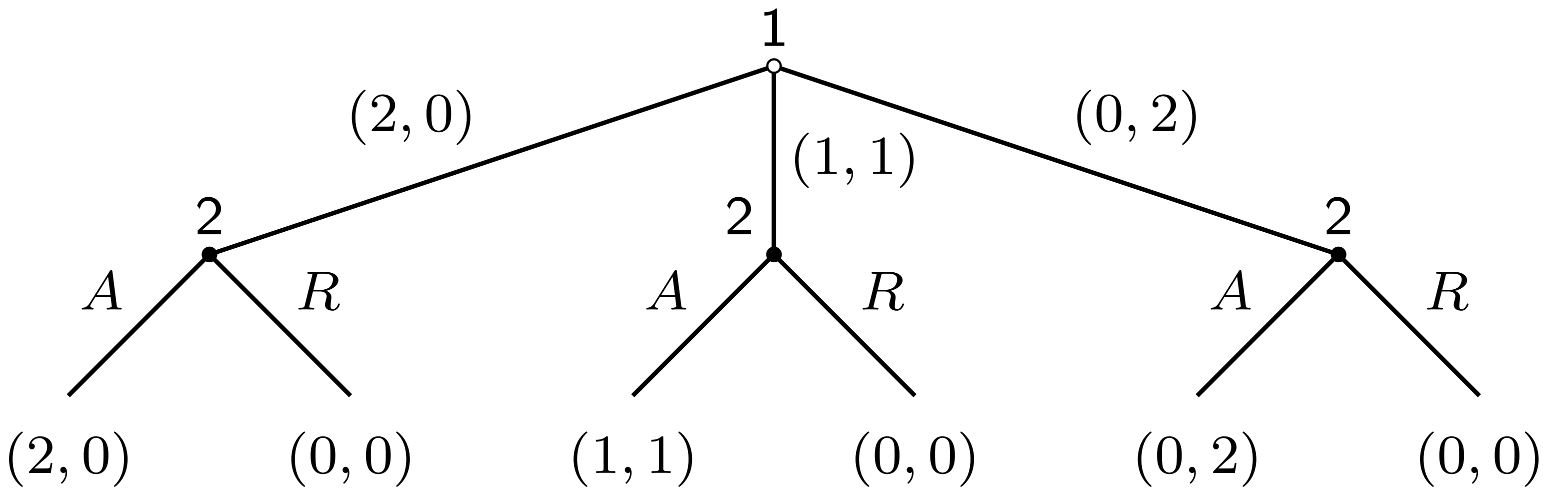

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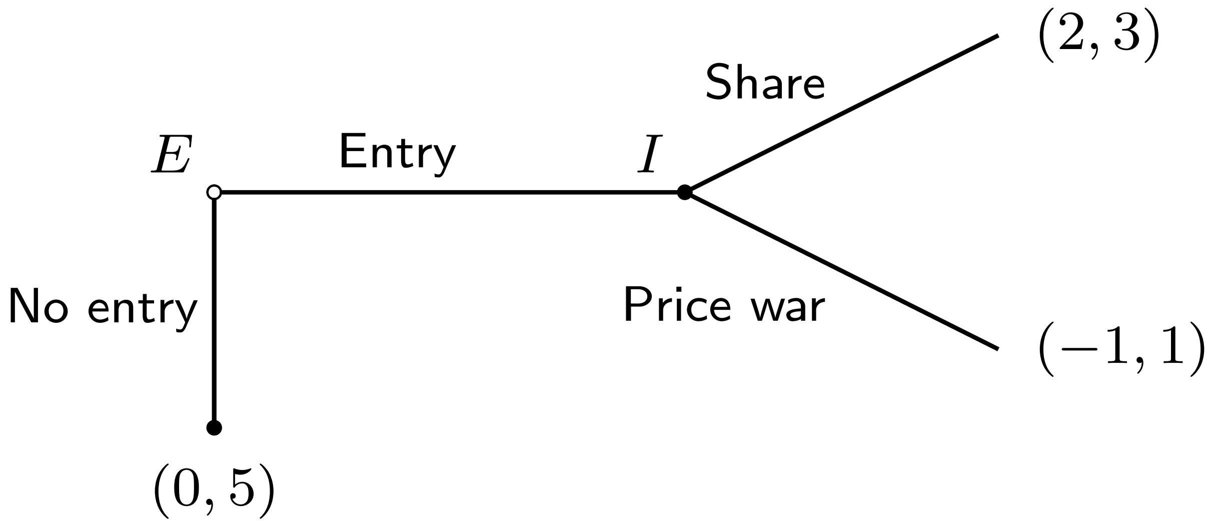

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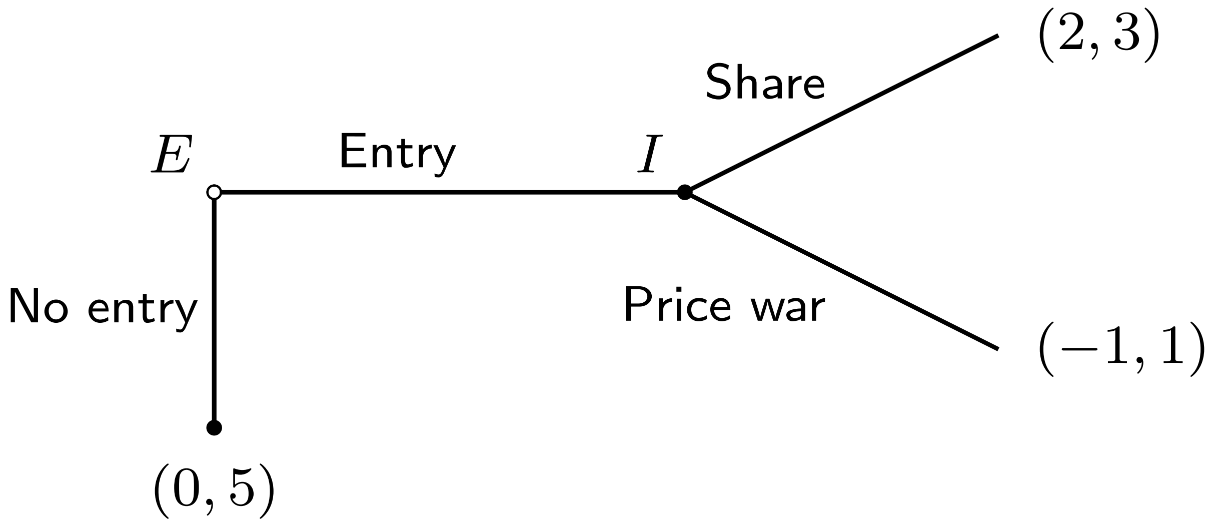

✍ Another example: owing a gun pdf

(Compare the simultaneous and the sequential game)

Perfect / Imperfect Information

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Otherwise, the game is of **imperfect information** (poker, Bertrand/Cournot duopoly, prisoner dilemma)

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by introducing a fictitious player, called Nature, who determines random events of the game (the states of Nature, including players' beliefs), with a common probability distribution

Particular case: Bayesian games

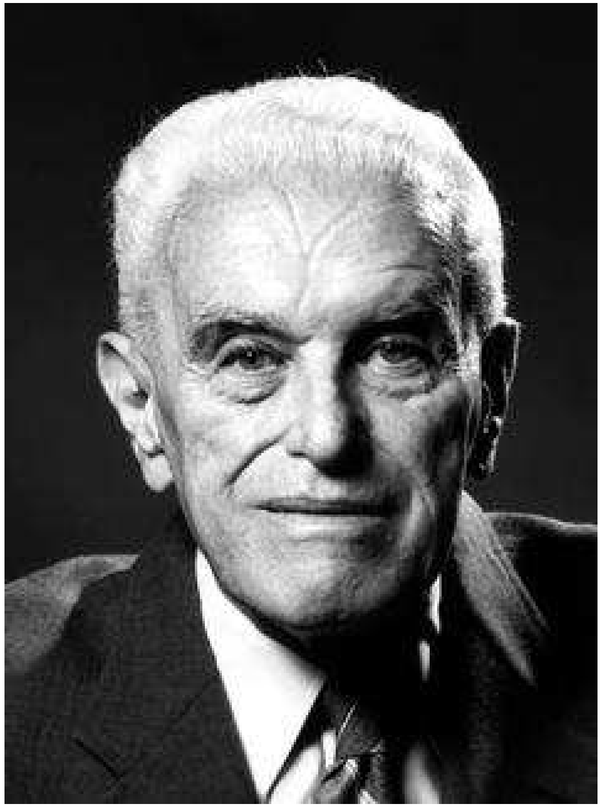

Figure 1: John C. Harsanyi (1920–2000)

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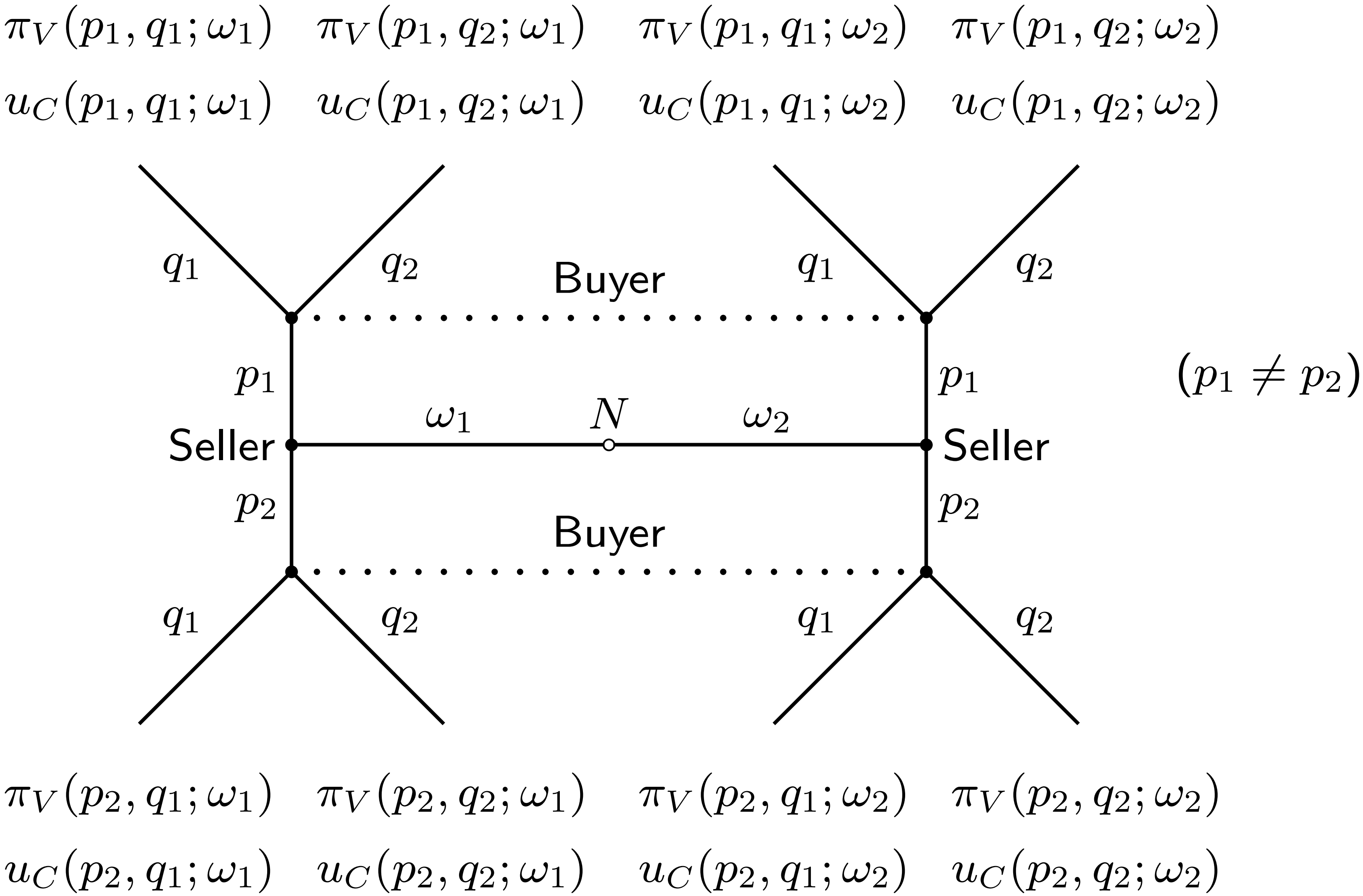
Player 1 (the informed player) is called the **sender** and player 2 (the uninformed player) is the **receiver**


$\pi_V(p_1, q_1; \omega_1)$ $\pi_V(p_1, q_2; \omega_1)$ $\pi_V(p_1, q_1; \omega_2)$ $\pi_V(p_1, q_2; \omega_2)$
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q_1 q_2 Buyer q_1 q_2
p_1 p_1 $(p_1 \neq p_2)$
ω_1 N ω_2
Seller Seller
p_2 p_2
Buyer
q_1 q_2 q_1 q_2
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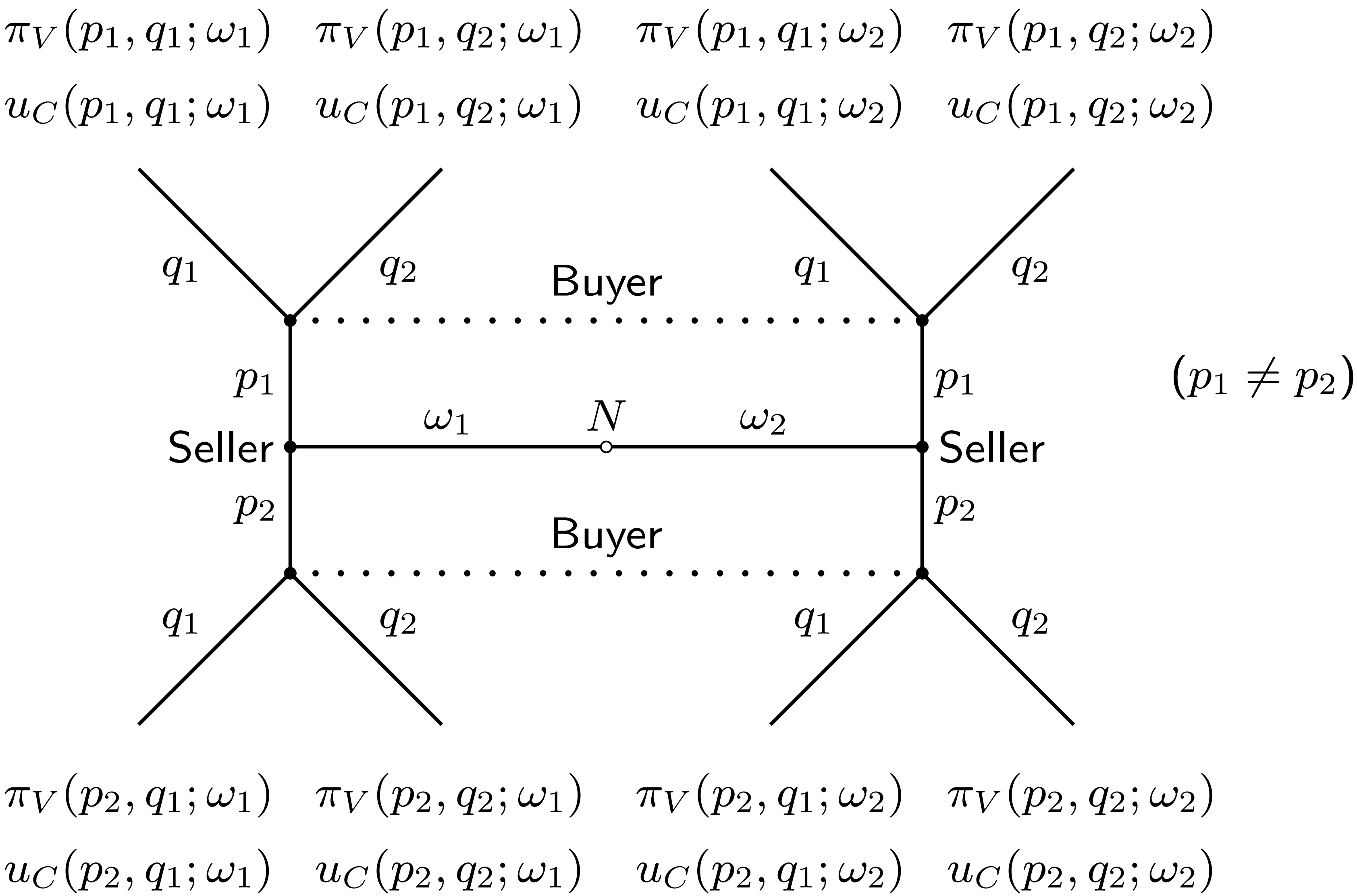

When players' payoff do not depend on the sender's action, the signaling game is called a **cheap talk game**

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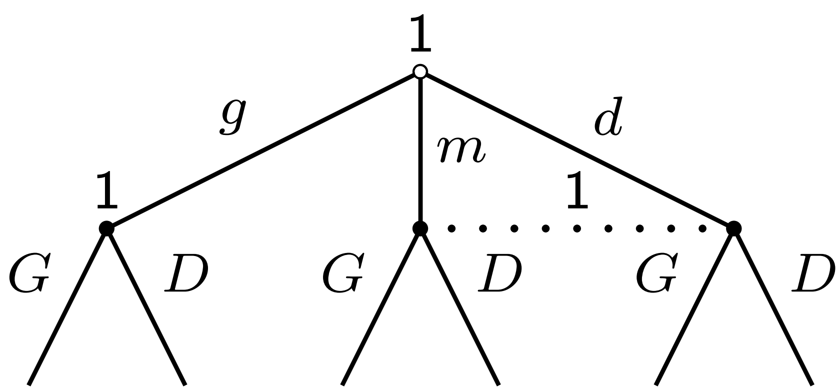
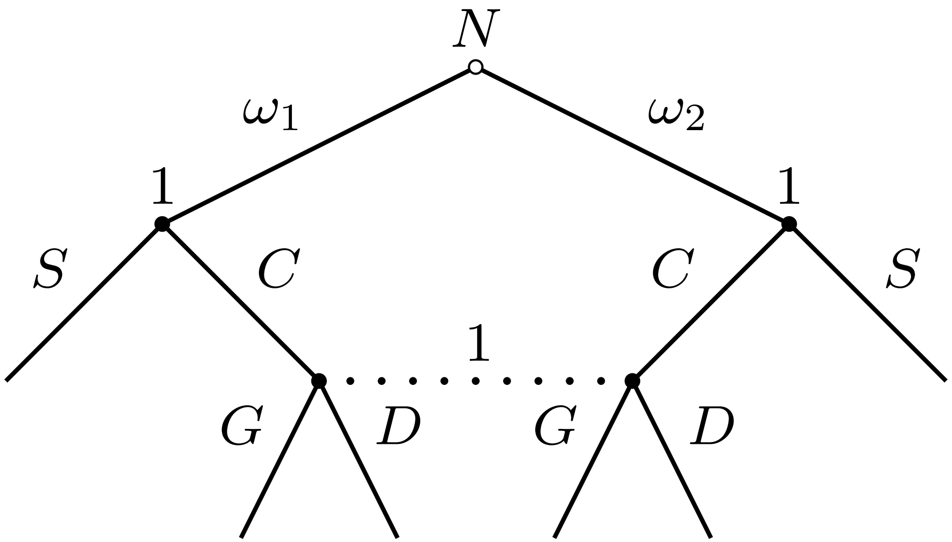
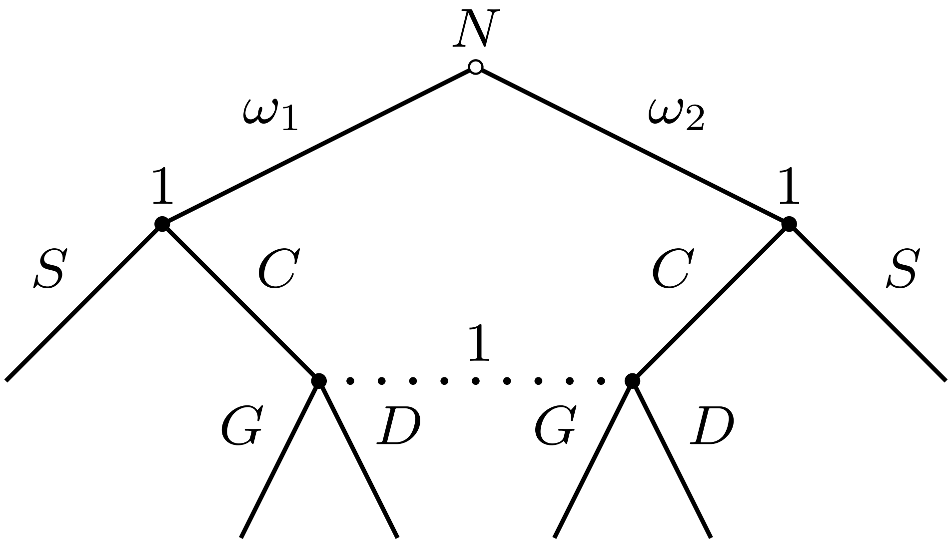




N
ω1
ω2
1
1
S
C
C
S
1
G
D
G
D


N
ω_1
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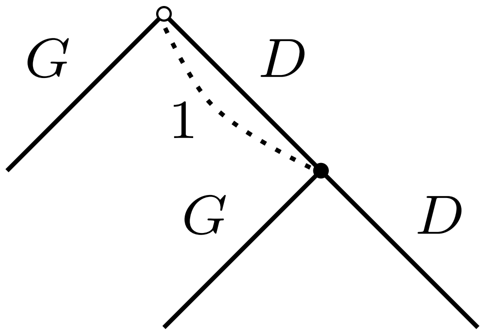
G
D
1
G
D

Strategies and Reduced Normal Form Game

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$$\begin{aligned} s_i : H_i &\longrightarrow A_i \\ h_i &\longmapsto a_i \in A(h_i) \end{aligned}$$

which associates to every information set $h_i \in H_i$ an action $a_i \in A(h_i)$, where $A(h_i)$ is the set of actions available at h_i

Strategy profile $+$ probability distribution over Ω

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➨

Probability distribution over terminal nodes

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$$\underbrace{\text{Expected utilities for every strategy profile}}_{\text{Normal form game}}$$

Example: Ultimatum Game (finite)

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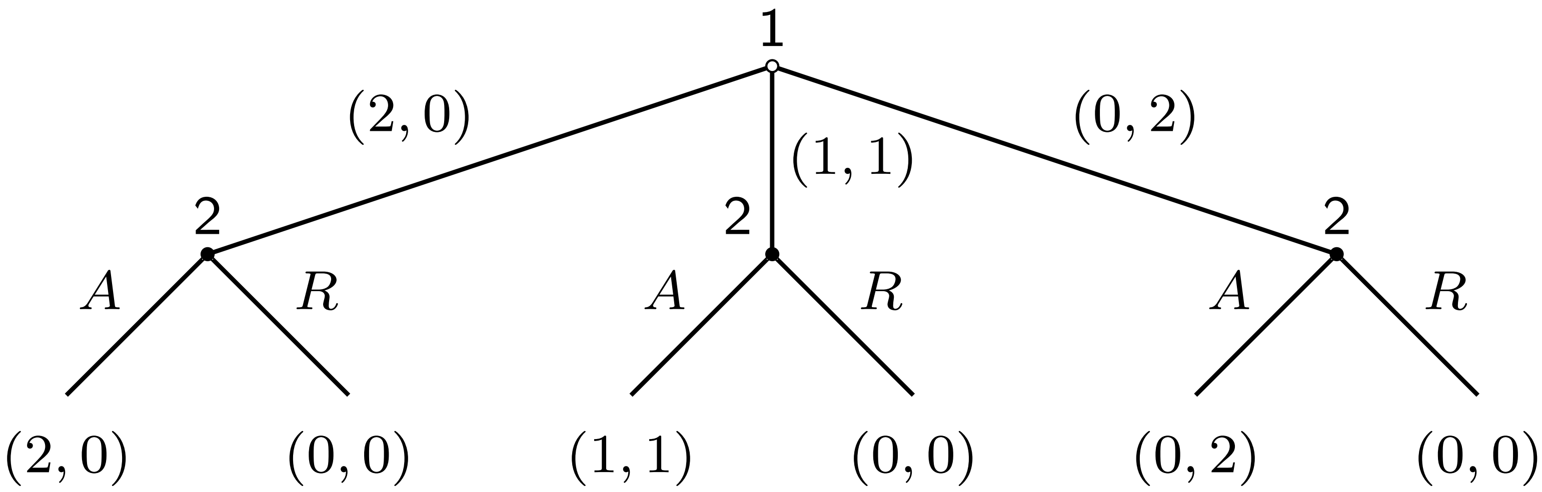

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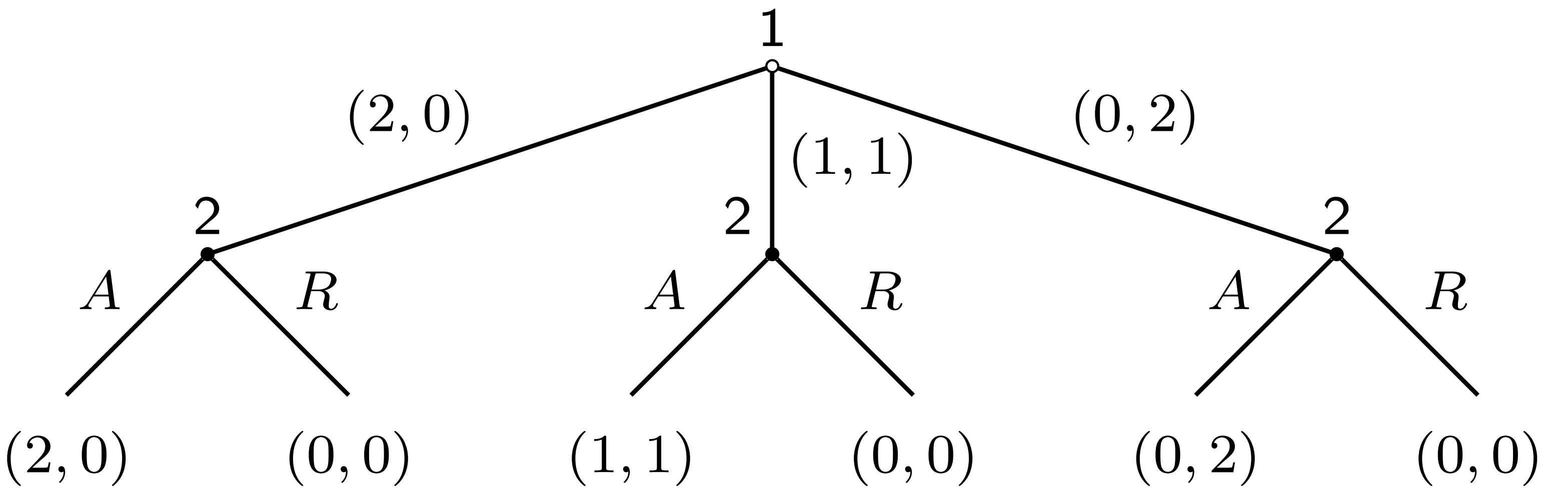

	AAA	RAA	ARA	AAR	RRA	RAR	ARR	RRR
$(2,0)$	$(2,0)$	$(0,0)$	$(2,0)$	$(2,0)$	$(0,0)$	$(0,0)$	$(2,0)$	$(0,0)$
$(1,1)$	$(1,1)$	$(1,1)$	$(0,0)$	$(1,1)$	$(0,0)$	$(1,1)$	$(0,0)$	$(0,0)$
$(0,2)$	$(0,2)$	$(0,2)$	$(0,2)$	$(0,0)$	$(0,2)$	$(0,0)$	$(0,0)$	$(0,0)$

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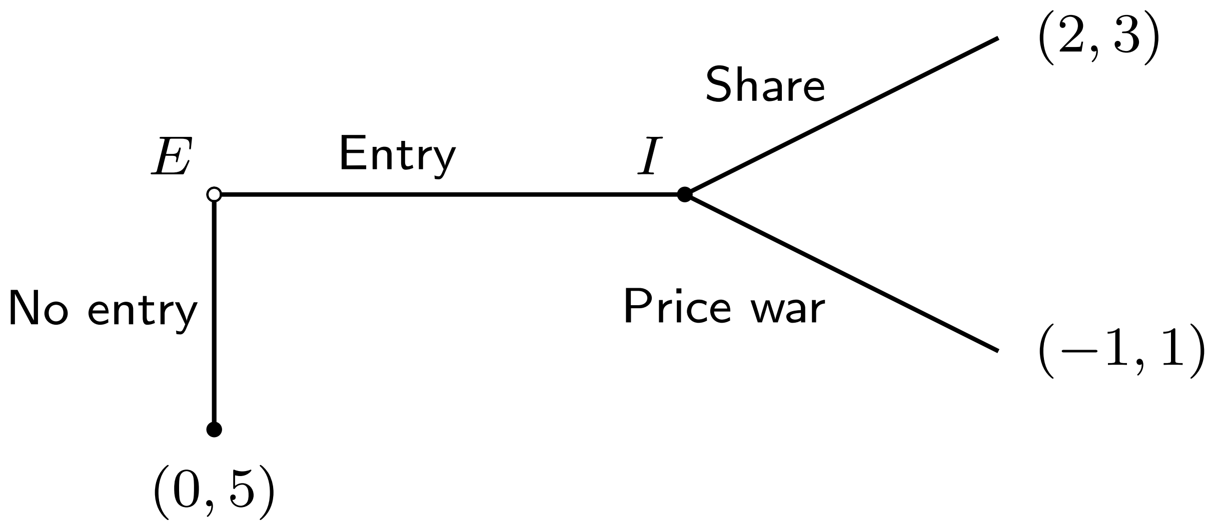

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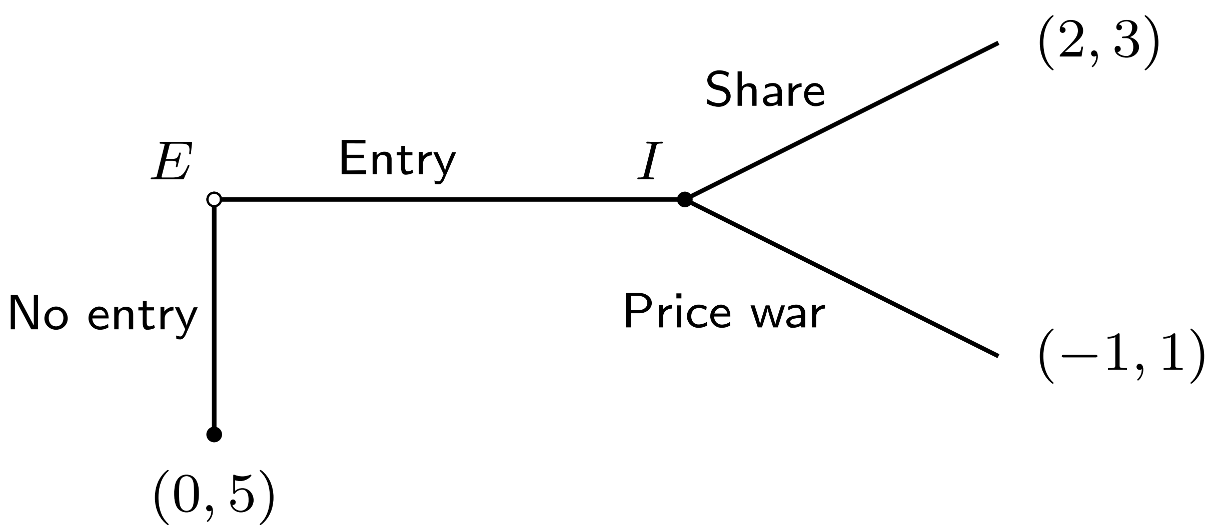

I

E		Share	Price war
	Entry	$2,3$	$-1,1$
	No entry	$0,5$	$0,5$

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- ✓ the value if the game is 0-sum

as in normal form games

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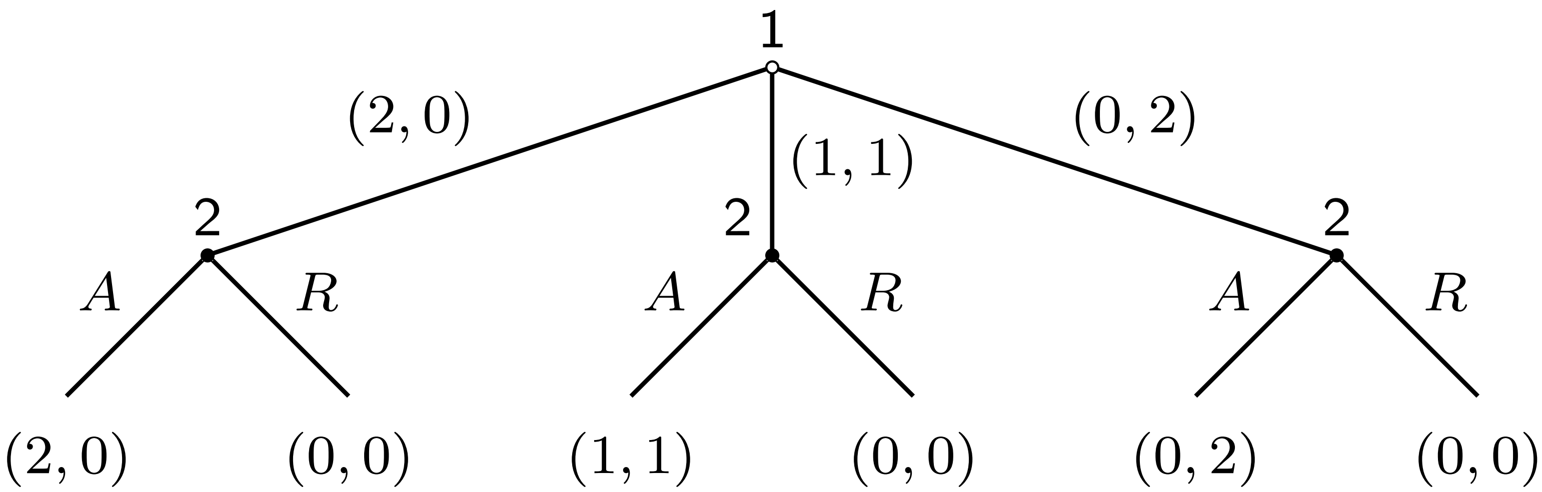
- Behavior strategy of player 2 : 3 probability distributions over $\{A, R\}$

A mixed strategy is outcome equivalent to a behavior strategy if whatever others' strategies, the two strategies generate the same probability distribution over terminal nodes

Example.

Example. In the ultimatum game



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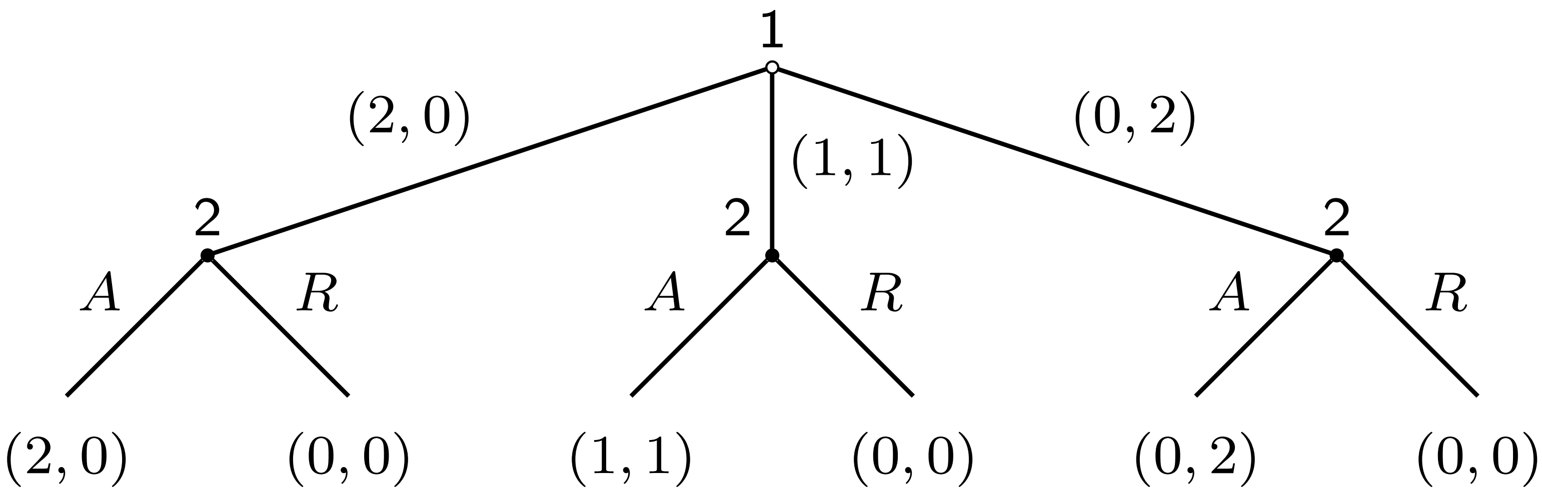

the mixed strategy $\sigma_2(AAA) = \sigma_2(ARA) = \sigma_2(AAR) = 1/3$ is equivalent to the behavior strategy $\beta_{h_2}(A) = 1,\ \beta_{h_2'}(A) = \beta_{h_2''}(A) = 2/3$, where h_2, h_2', h_2'' are the information sets of player 2

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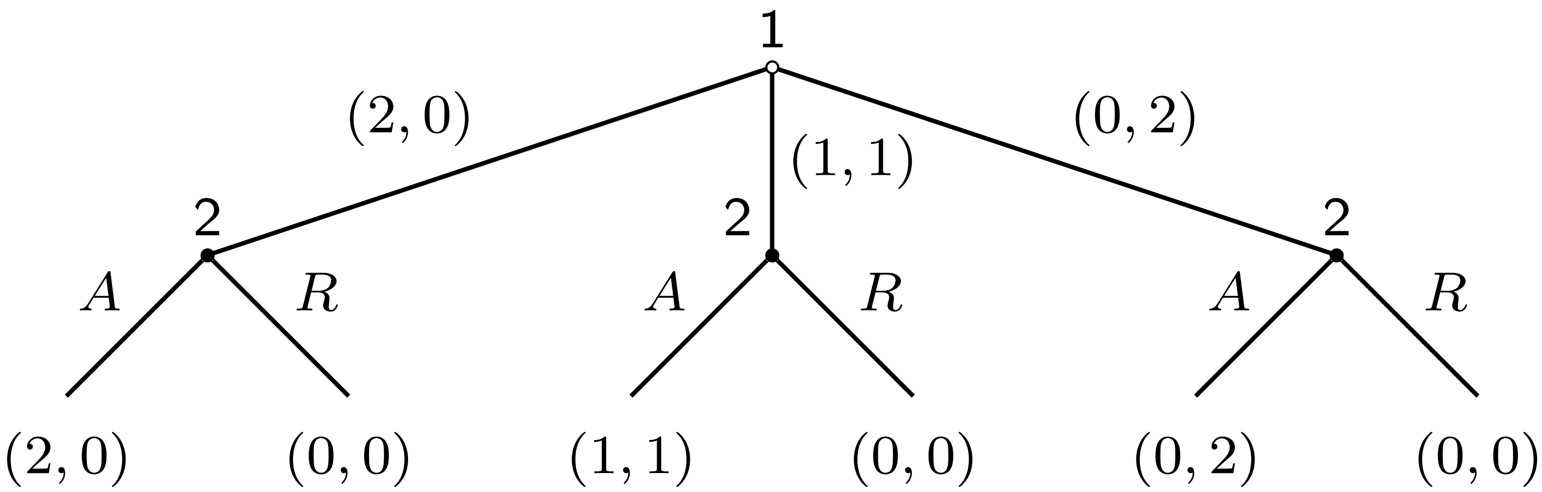

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Remark: Several mixed strategies are equivalent to β_2 (for example, $\sigma_2(AAA) = 2/3$ and $\sigma_2(ARR) = 1/3$)

Example.

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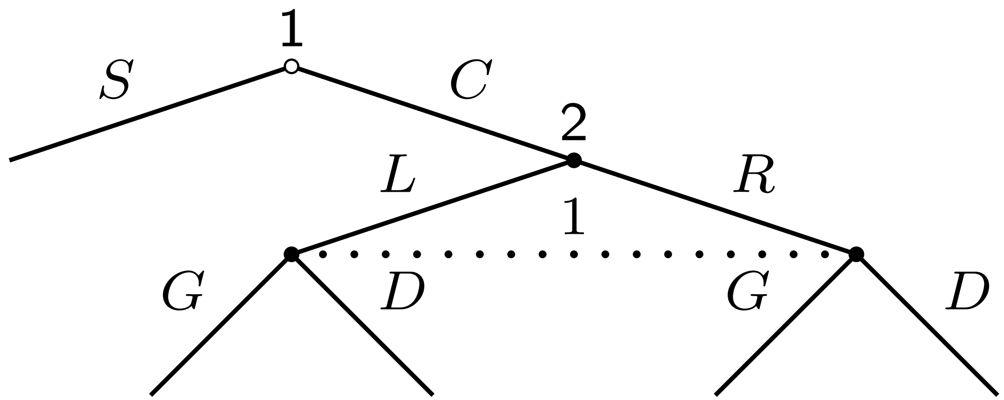

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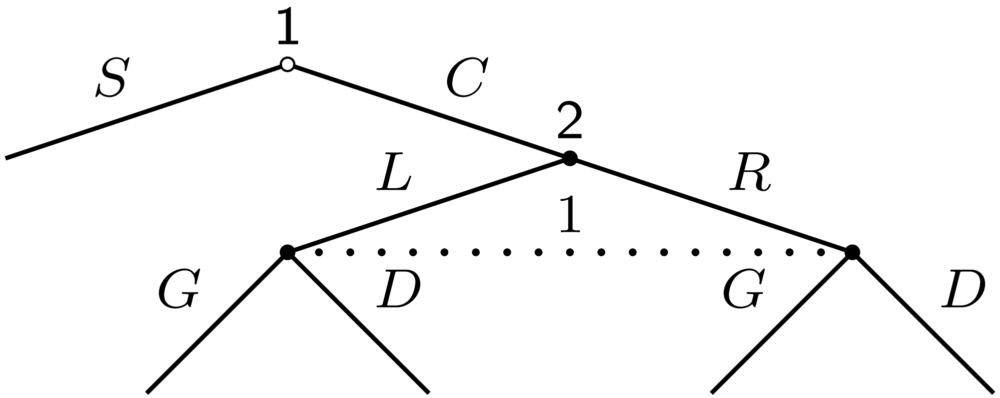
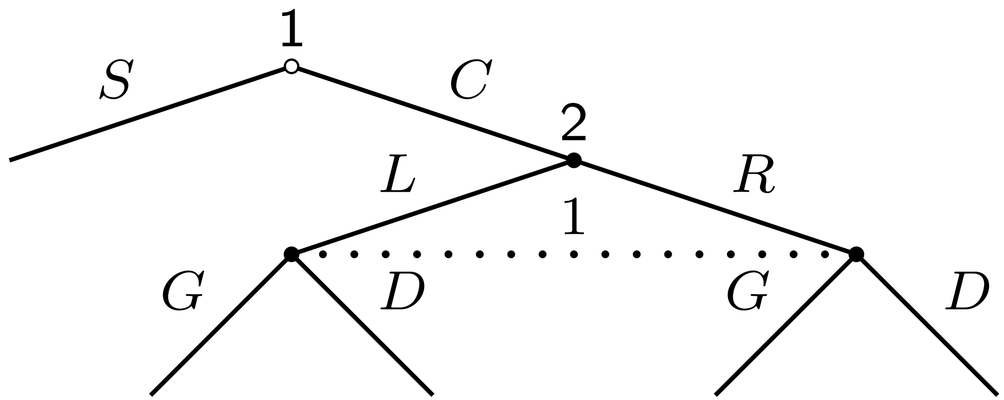


The mixed strategy

$$\sigma_1(S, D) = 0.4,\ \sigma_1(S, G) = 0.1,\ \sigma_1(C, D) = 0.5$$

Example.



The mixed strategy

$$\sigma_1(S,D) = 0.4,\ \sigma_1(S,G) = 0.1,\ \sigma_1(C,D) = 0.5$$

is equivalent to the behavior strategy of player 1 that consists in playing S and C with probability 1/2, and D with probability 1

Proposition. (Kuhn, 1953)

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In every finite extensive form game with perfect memory, for every mixed strategy (behavior strategy, resp.) there exists an outcome equivalent behavior strategy (mixed strategy, resp.)

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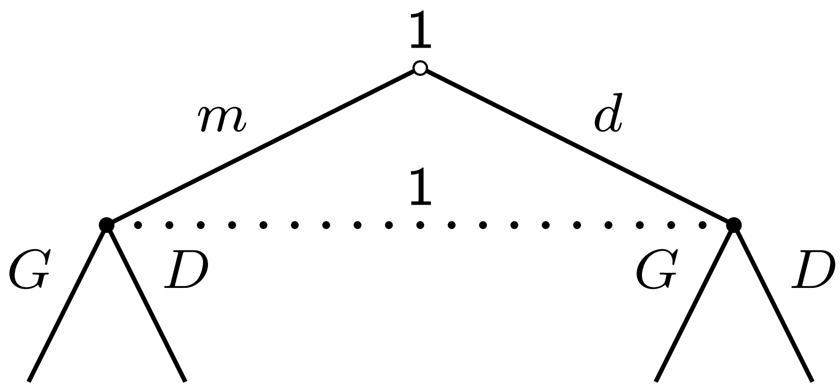

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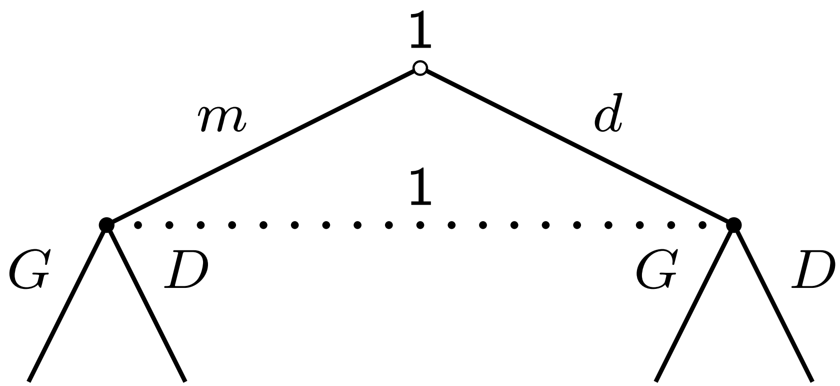
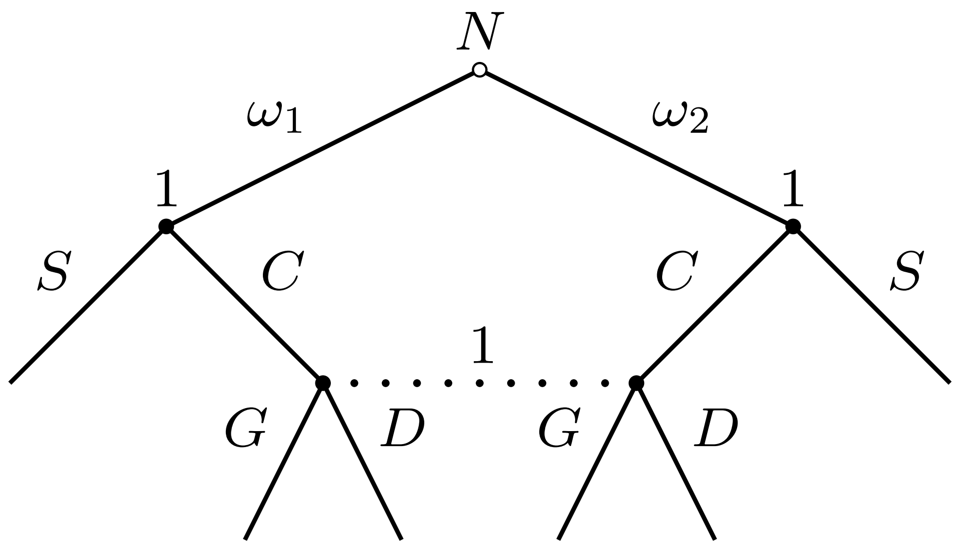
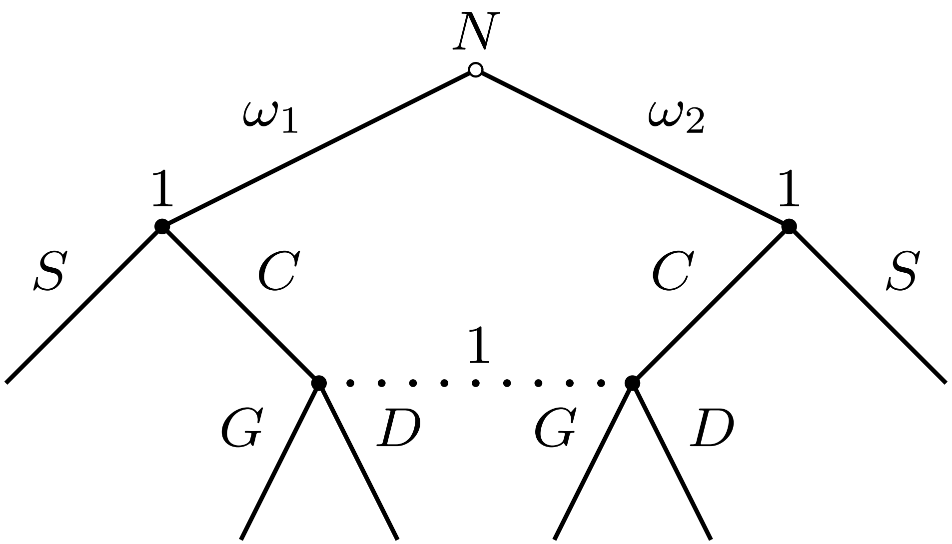
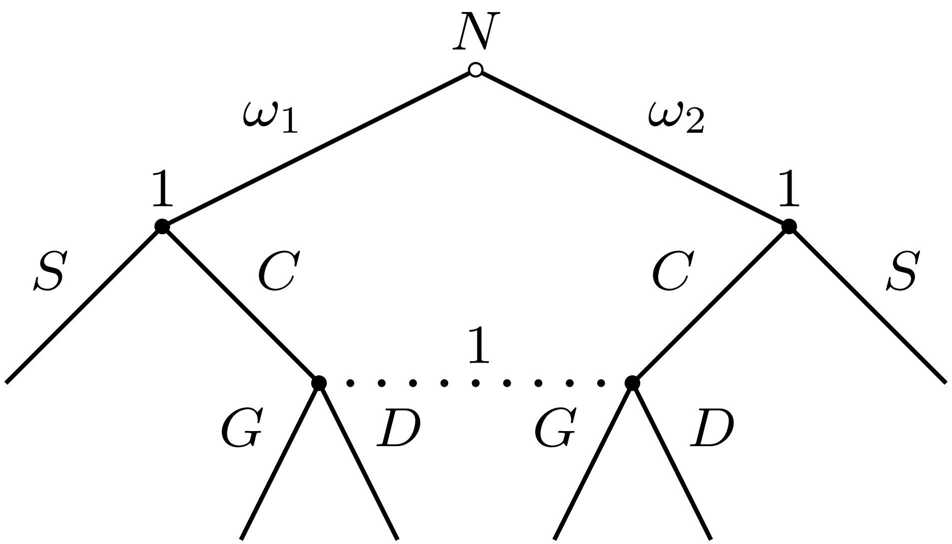


➥ The mixed strategy $\sigma_1(m, G) = \sigma_1(d, D) = 1/2$ has no equivalent behavior strategy


N
ω_1
ω_2
1
1
S
C
C
S
1
G
D
G
D



➥ The mixed strategy $\sigma_1(C, C, G) = \sigma_1(C, C, D) = 1/2$ has an equivalent behavior strategy ($C \mid \omega_1$, $C \mid \omega_2$, $\frac{1}{2}G + \frac{1}{2}D \mid C$)



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➥ But the mixed strategy $\sigma_1(C, C, G) = \sigma_1(C, S, D) = 1/2$ has no equivalent behavior strategy

Incredible Threats

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Some Nash equilibria are not “adequate” if players are fully rational because they rely on irrational behavior (incredible threats) off the equilibrium path

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Examples: image image

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Examples: image image

- Entry game: (No entry, **price war**)

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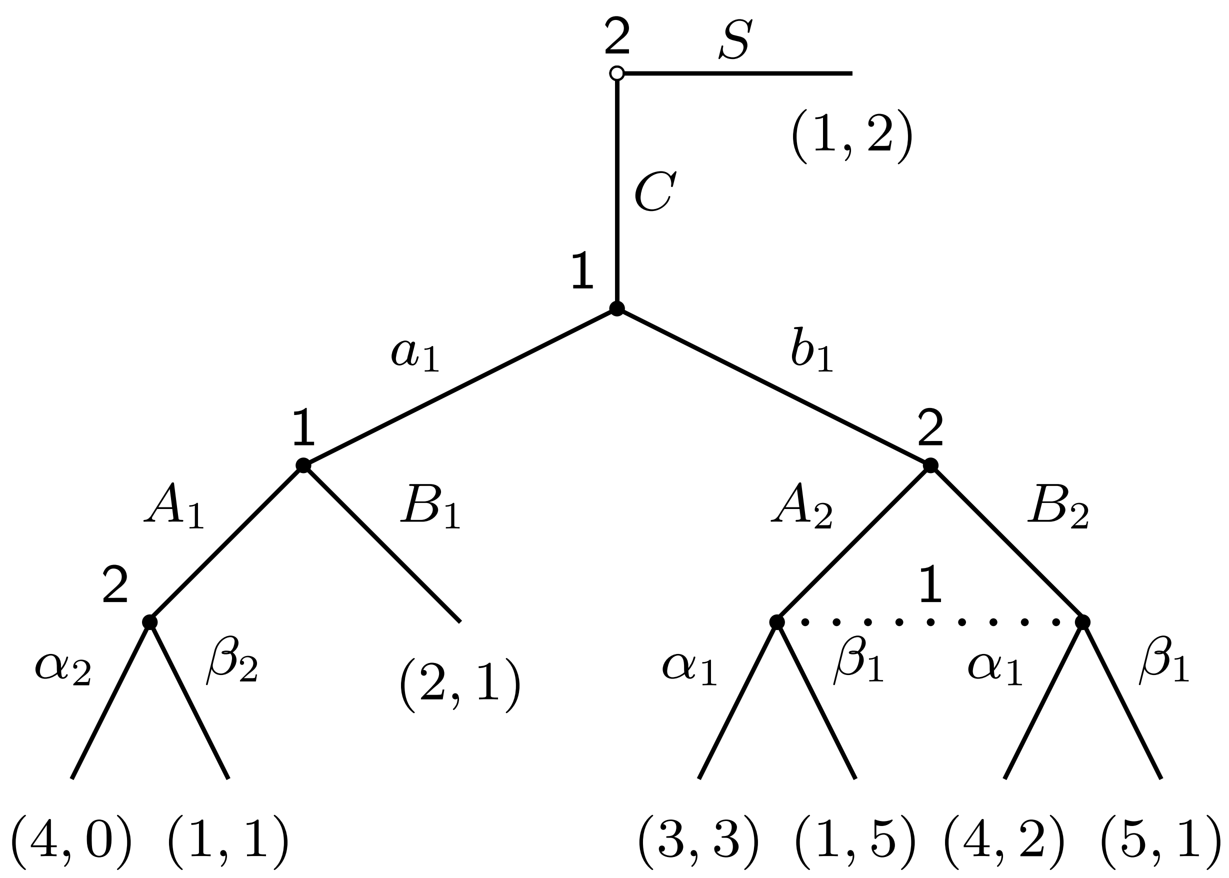
Examples: image image

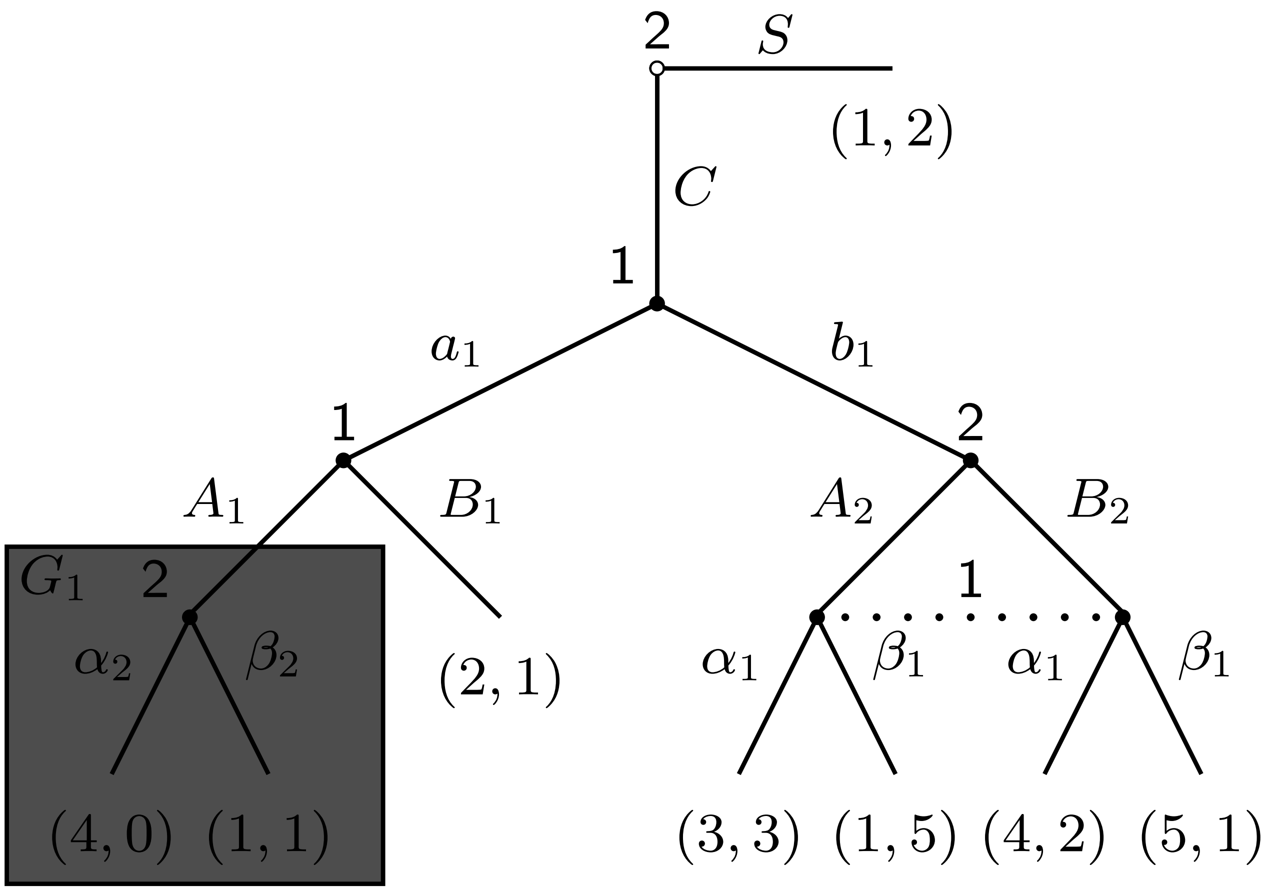
- Entry game: (No entry, **price war**)
- Ultimatum game: $((0, 2),\ R\boldsymbol{R}A)$

Subgames

Subgames

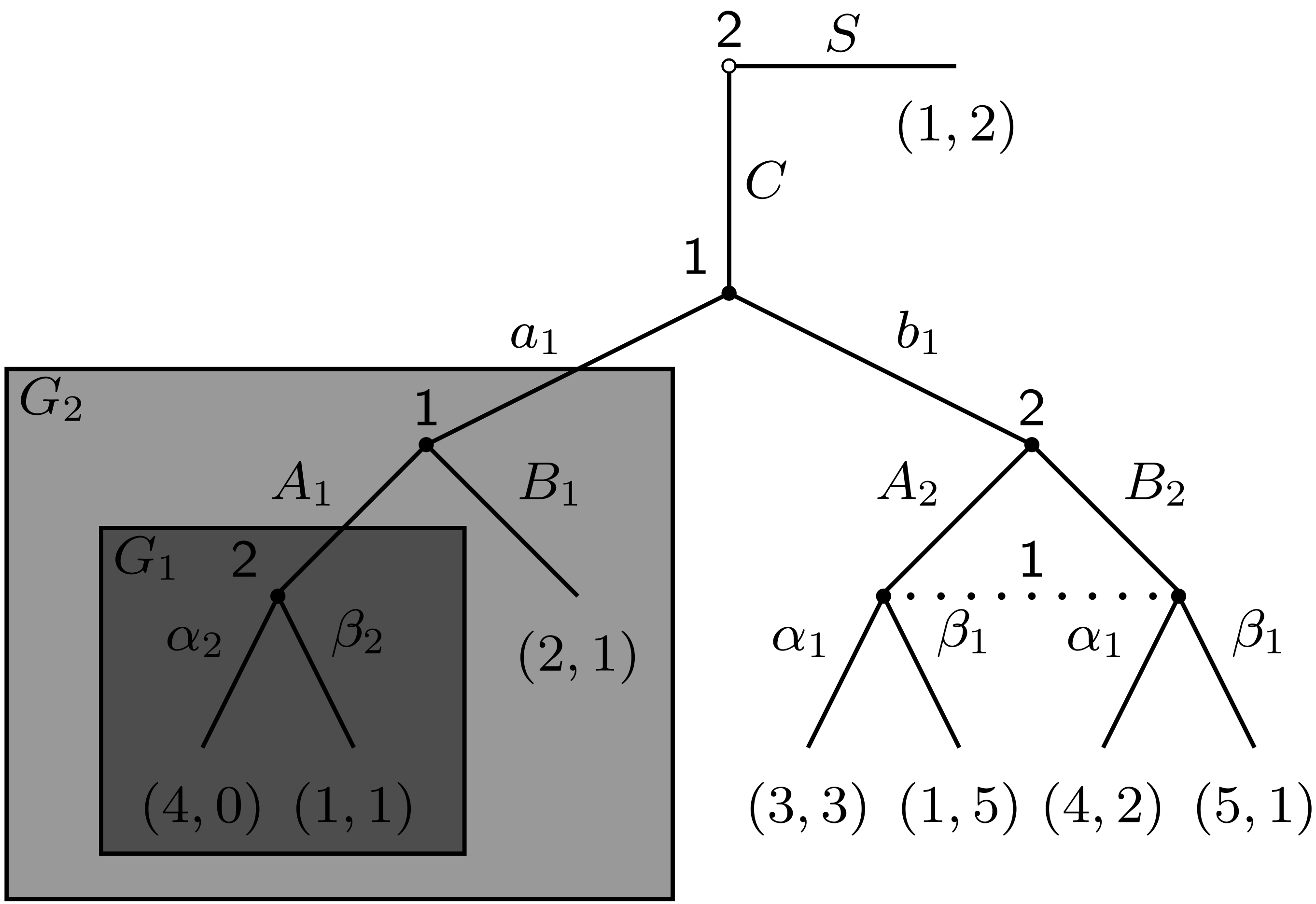


Subgames



2
S
$(1,2)$
C
1
a_1
b_1
1
2
A_1
B_1
A_2
B_2
G_1
2
1
α_2
β_2
$(2,1)$
α_1
β_1
α_1
β_1
$(4,0)$
$(1,1)$
$(3,3)$
$(1,5)$
$(4,2)$
$(5,1)$

Subgames



2
S
$(1,2)$
C
1
a_1
b_1
G_2
1
2
A_1
B_1
A_2
B_2
G_1
2
1
α_2
β_2
$(2,1)$
α_1
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α_1
β_1
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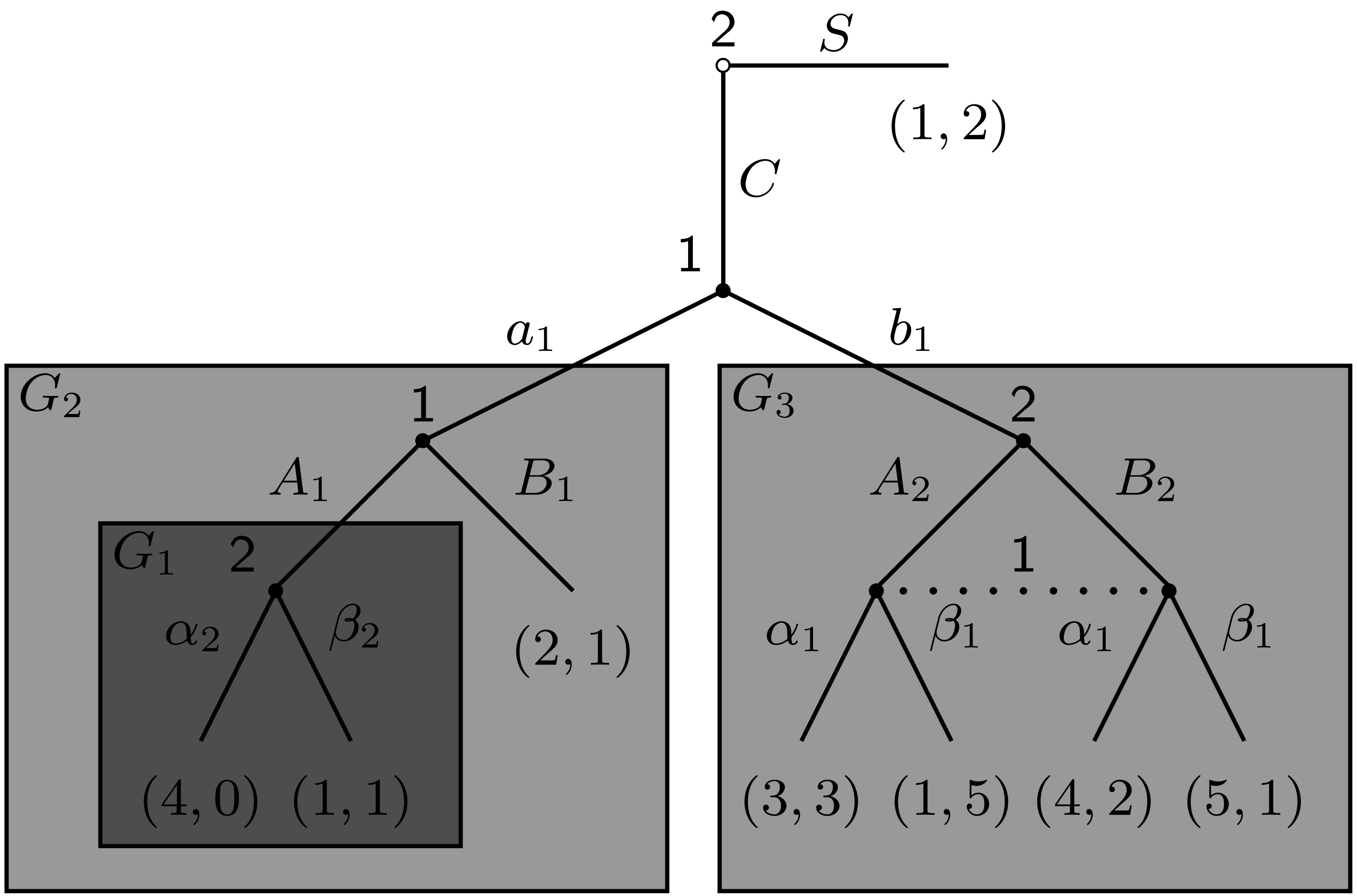
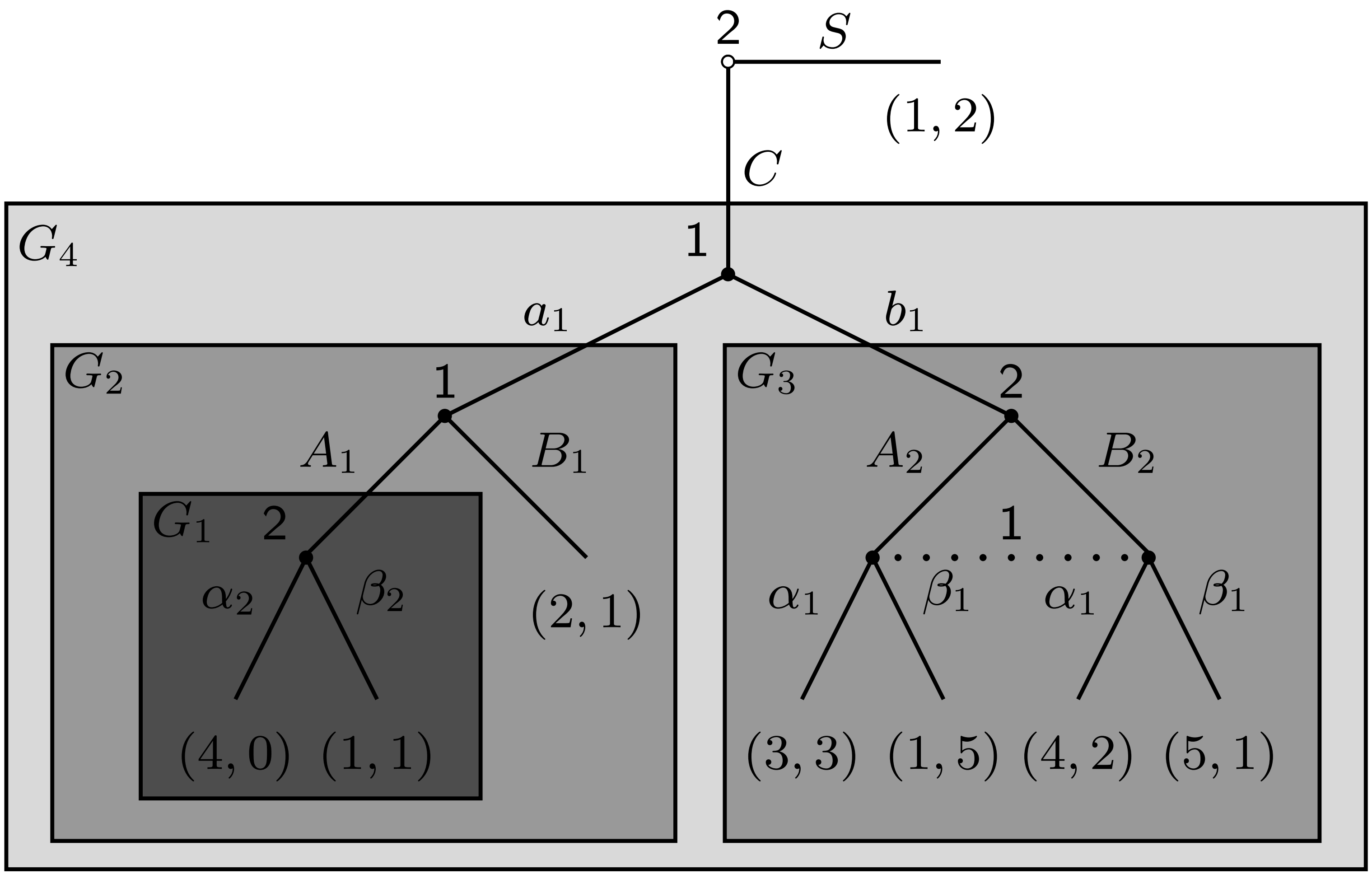


2
S
$(1, 2)$
C
1
a_1
b_1
G_2
1
A_1
B_1
G_1
2
α_2
β_2
$(2, 1)$
$(4, 0)$ $(1, 1)$
G_3
2
A_2
B_2
1
α_1 β_1 α_1 β_1
$(3, 3)$ $(1, 5)$ $(4, 2)$ $(5, 1)$

Subgames



2
S
(1, 2)
C
G4
1
a1
b1
G2
1
G3
2
A1
B1
A2
B2
G1
2
1
α2
β2
(2, 1)
α1
β1
α1
β1
(4, 0)
(1, 1)
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(1, 5)
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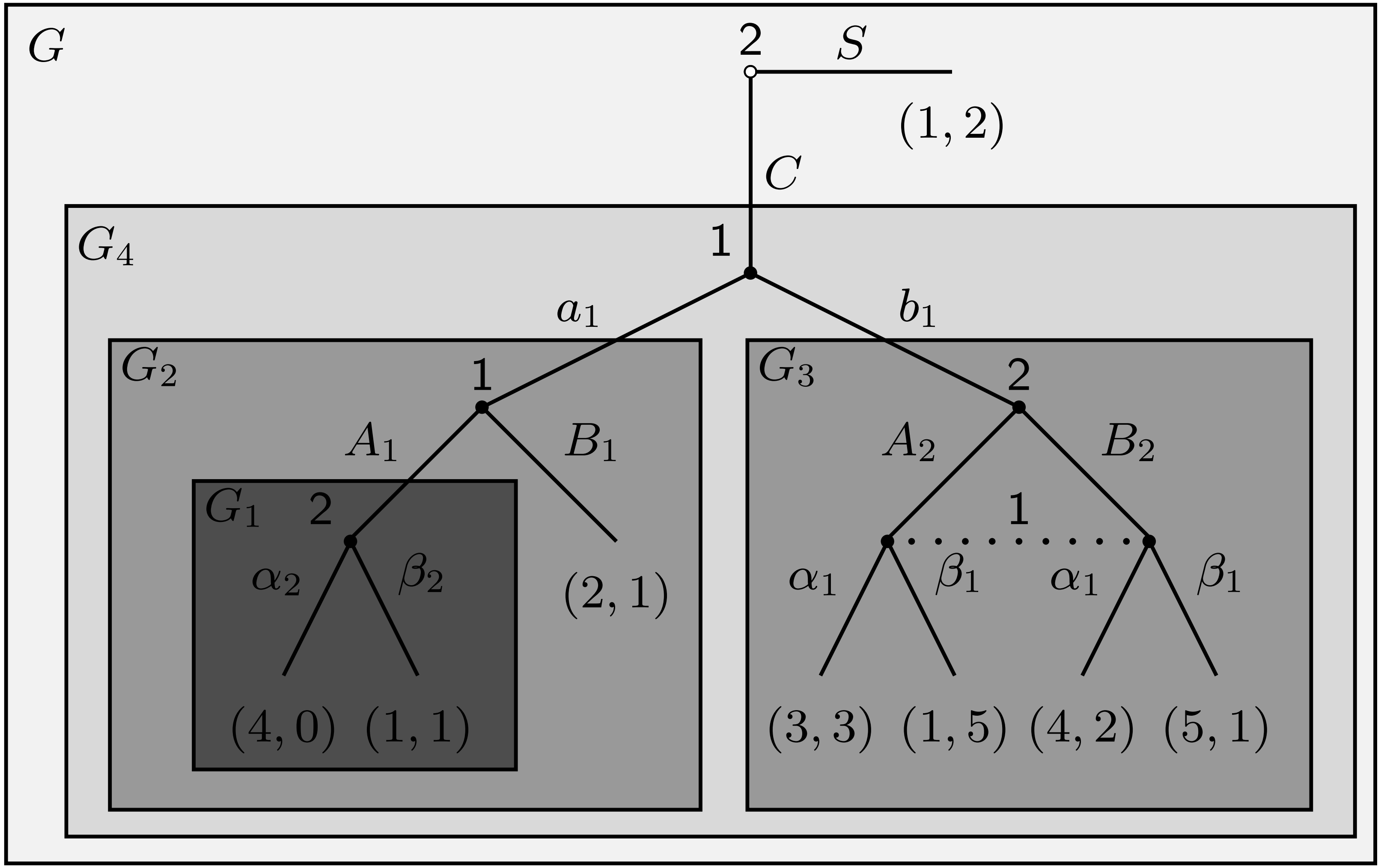
G
2
S
(1, 2)
C
G4
1
a1
b1
G2
1
G3
2
A1
B1
A2
B2
G1
2
1
α2
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✍ Subgames in previous examples?

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Definition. (Selten, 1965)

A subgame perfect Nash equilibrium (SPNE) is a profile of strategies such that in each subgame the induced strategy profile is a Nash equilibrium of that subgame

Figure 2: Reinhard Selten (1930–)

Remarks.

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Proposition.

Every finite extensive form game has at least one subgame perfect equilibrium

Backward Induction

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Solve the game starting from the end: first find the NE of the smallest subgames

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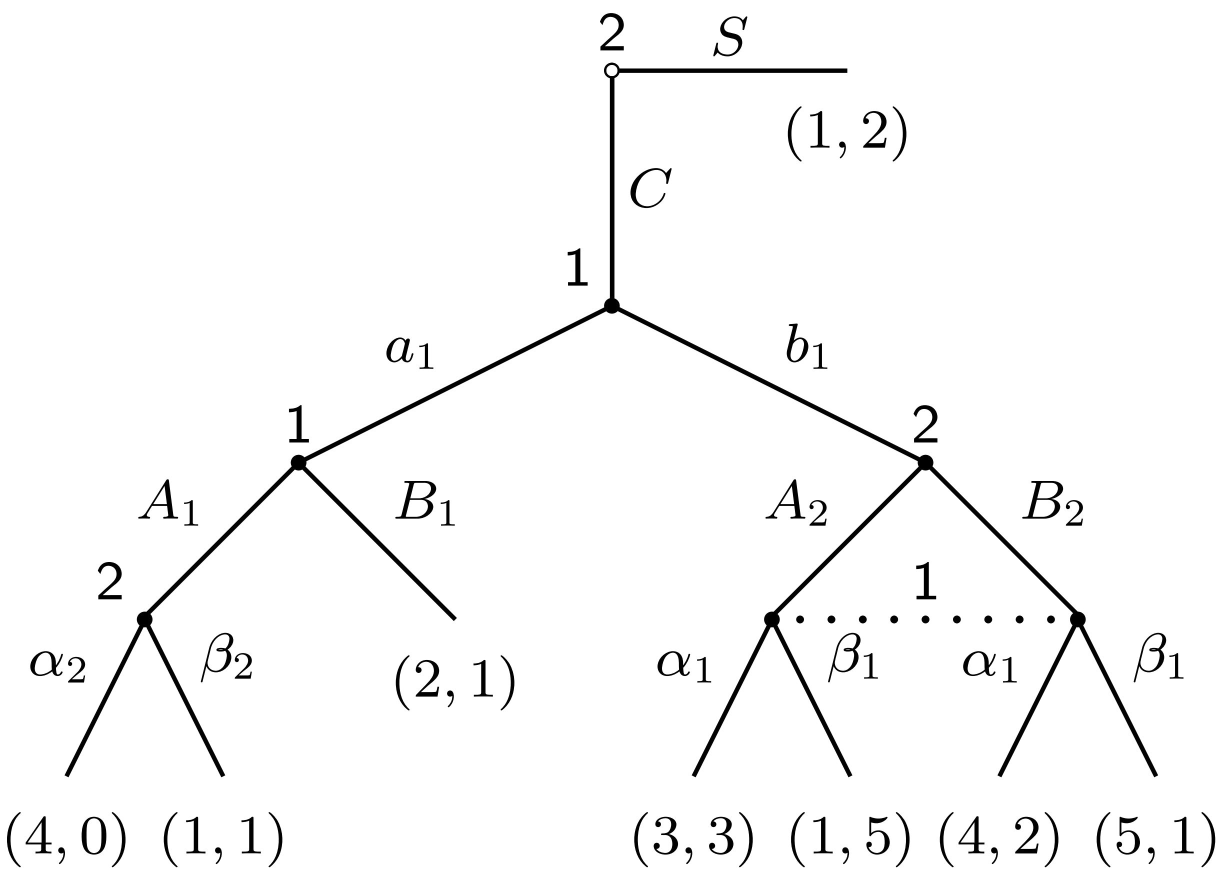

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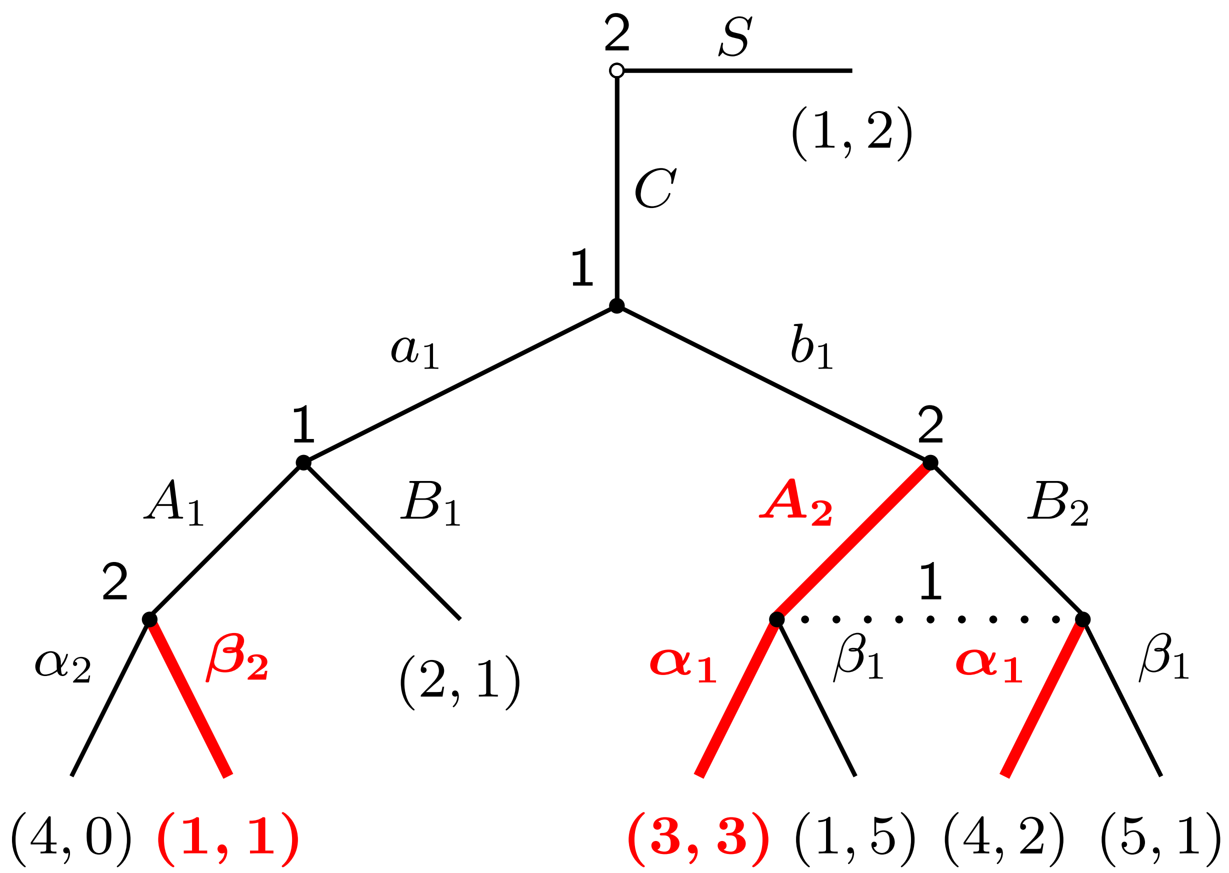

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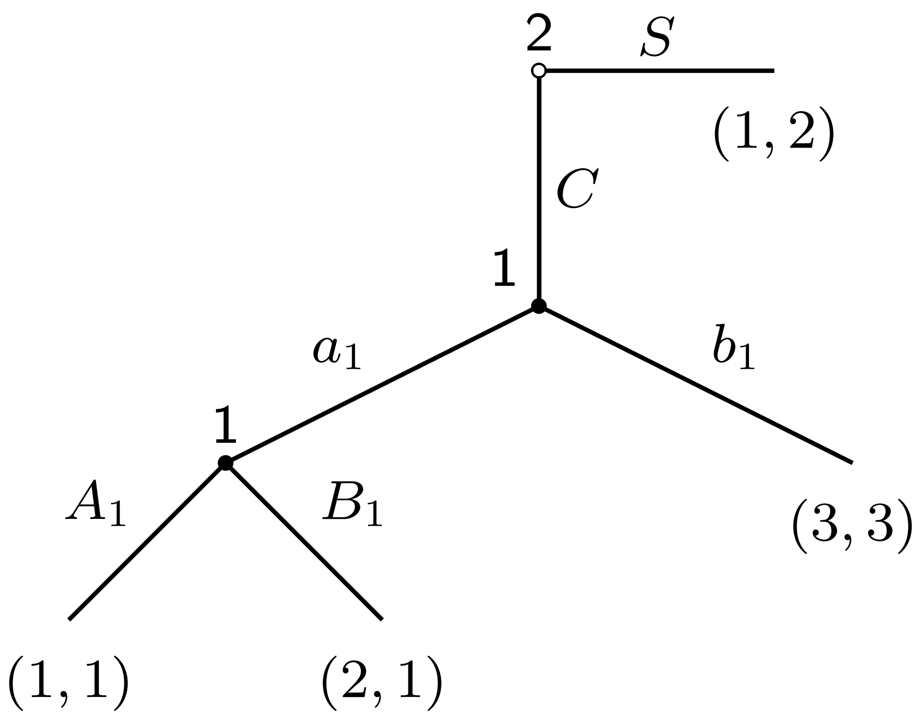

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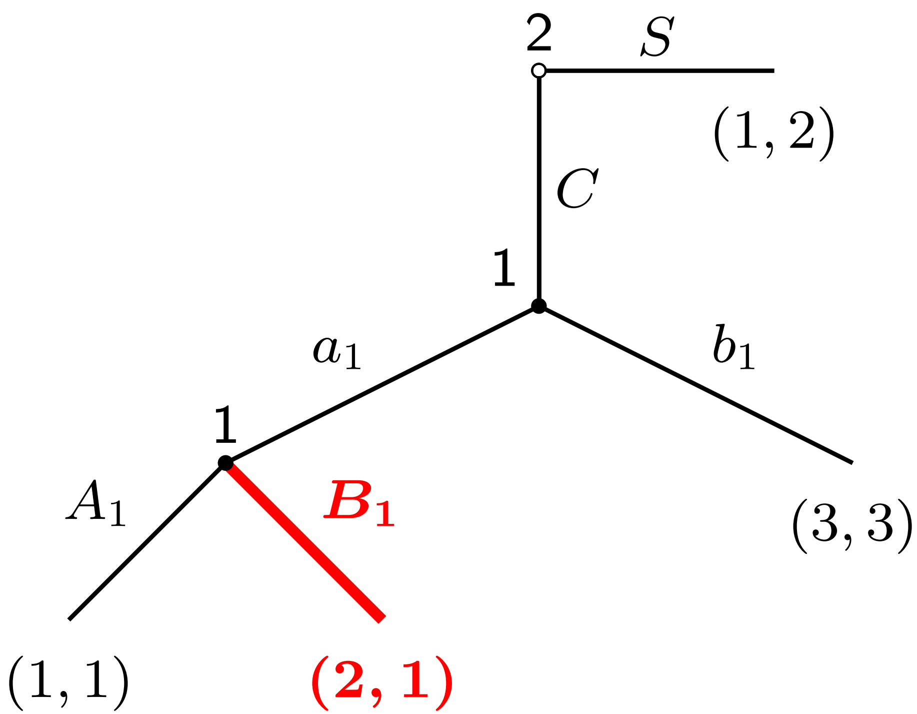

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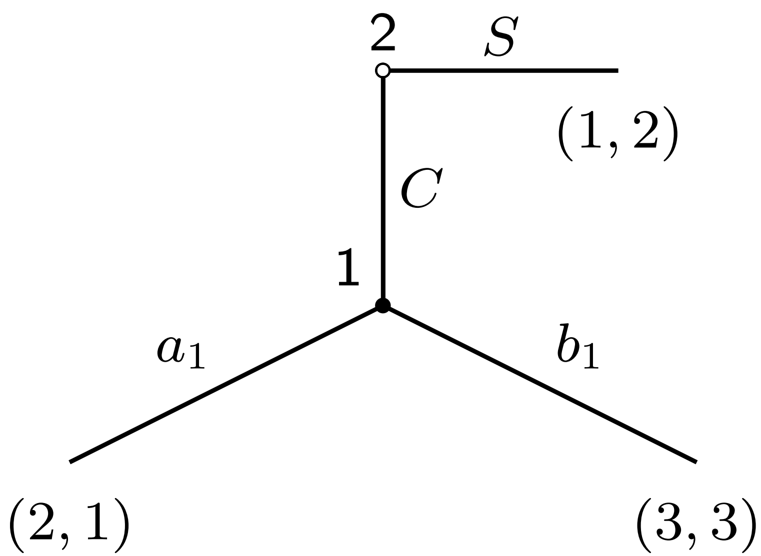

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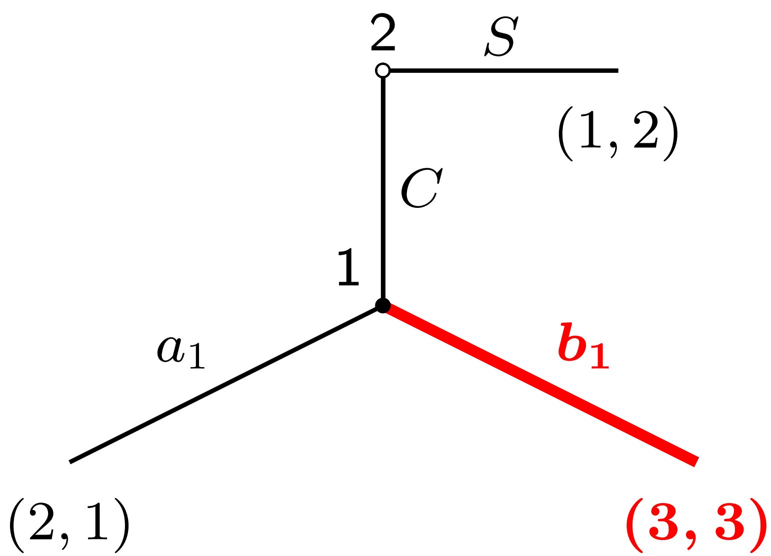

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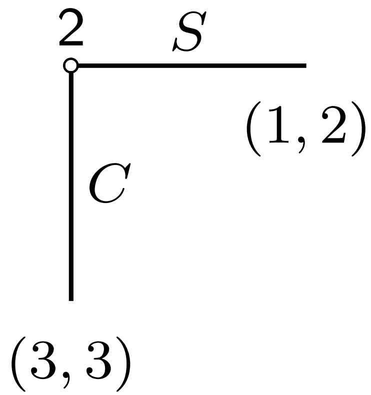

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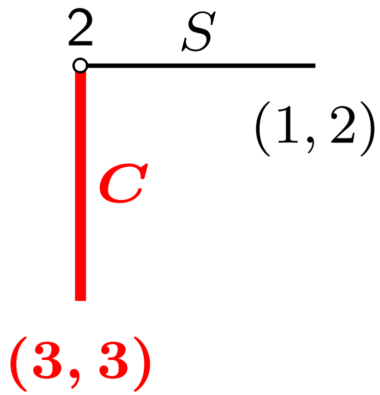

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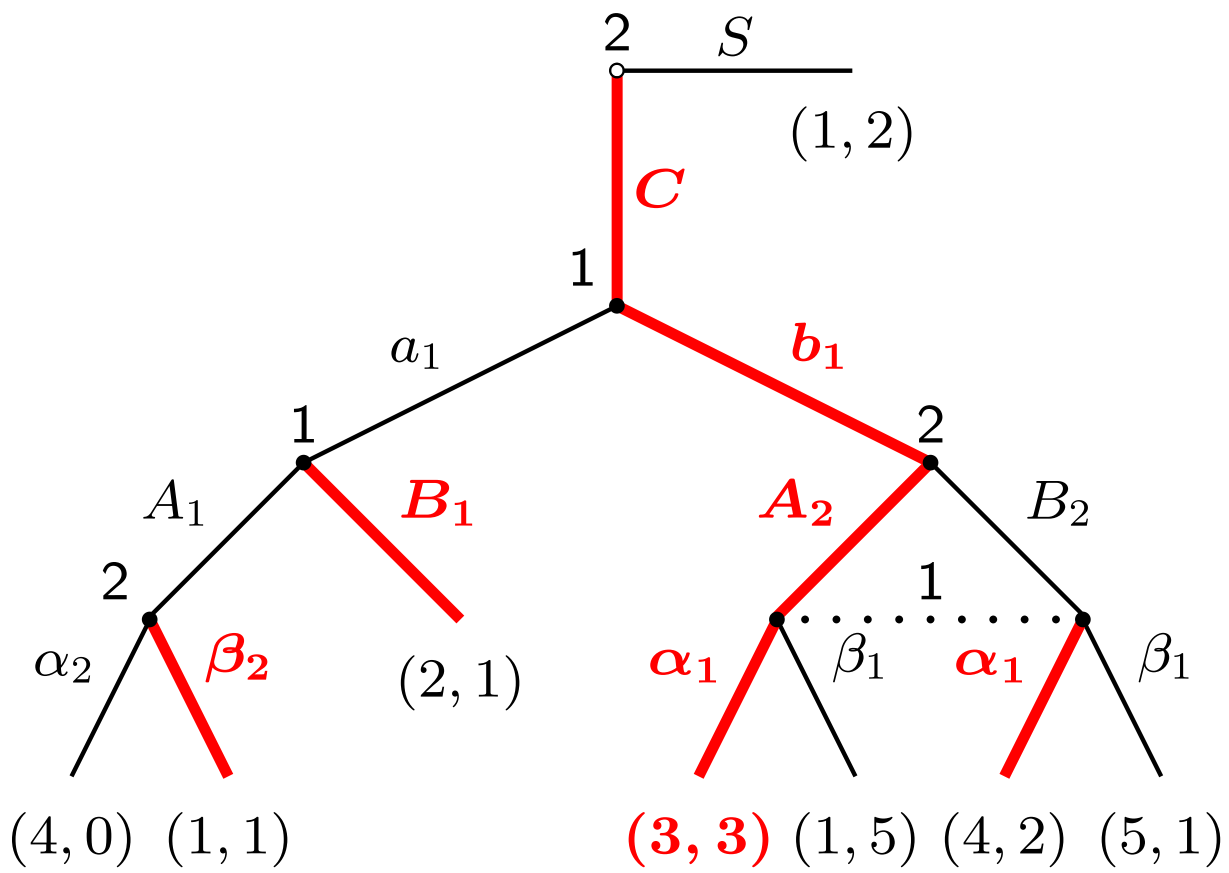

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Ultimatum game.

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Ultimatum game. Two SPNE in pure strategies:

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Remarks.

☞ The set of actions at every information must be finite: $A = [0,1)$ and $u_i(a) = a$ implies no SPNE

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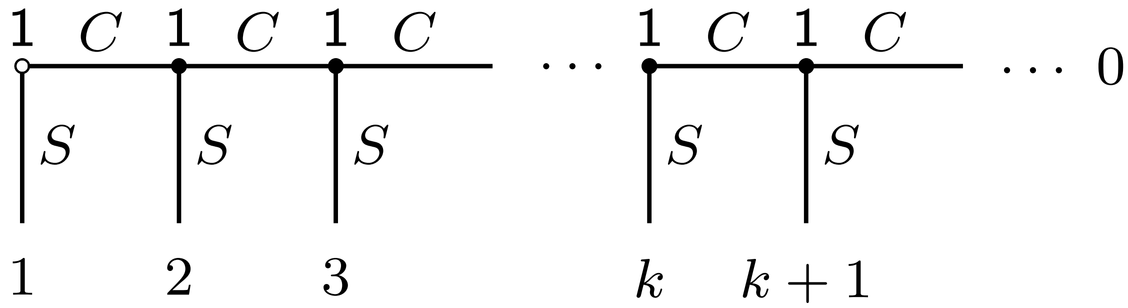

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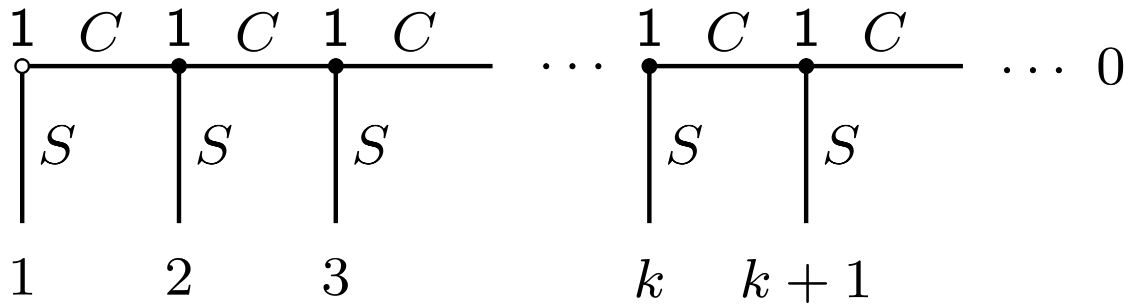

✍ Example to analyze: "winning without knowing how" pdf

Example. Incredible threat / commitment

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Army 1 of country 1 wants to attack army 2 of country 2 which is on an island between the two countries. If army 1 attacks then army 2 can choose between fighting and retreating using the bridge between the island and country 2. Each army prefers getting the island instead of letting it to its opponent, but the worst outcome is war

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Consider the initial situation again

✍ If decisions are simultaneous, what kind of game is it? (if the island turns out to be non-occupied, consider intermediate payoffs between being alone on the island and letting it to the enemy)

Stackelberg Duopoly

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Firm $i = 1, 2$ produces q_i with zero fixed cost and constant marginal cost $\lambda > 0$

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Profit of firm i :

$$u_i(q_1, q_2) = p(q_1 + q_2)\, q_i - \lambda\, q_i = q_i(a - \lambda - (q_1 + q_2))$$

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Firm 2's strategy: **function** $q_2^*(q_1)$

Backward induction solution.

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$$q_2^*(q_1) = \mathrm{BR}_2(q_1) = \arg\max_{q_2} u_1(q_1, q_2) = \frac{a - \lambda - q_1}{2}$$

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Optimal production of firm 1 given firm 2's response ➟ maximize

$$u_1(q_1, q_2^*(q_1)) = q_1(a - \lambda - (q_1 + q_2^*(q_1))) = \frac{1}{2} q_1(a - \lambda - q_1)$$

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i.e., $q_1^* = \frac{a-\lambda}{2} \Rightarrow q_2^*(q_1^*) = \frac{a-\lambda}{4}$

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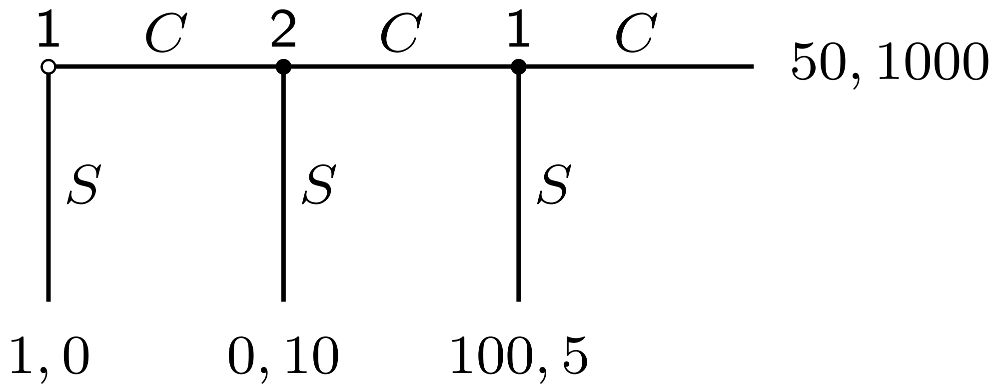
i.e., $q_1^* = \frac{a-\lambda}{2} \Rightarrow q_2^*(q_1^*) = \frac{a-\lambda}{4}$

	Cournot		Stackelberg (firm 1 leader)	
Firm 1	$q_1 = \frac{a-\lambda}{3}$	$u_1 = \frac{(a-\lambda)^2}{9}$	$q_1 = \frac{a-\lambda}{2}$	$u_1 = \frac{(a-\lambda)^2}{8}$
Firm 2	$q_2 = \frac{a-\lambda}{3}$	$u_2 = \frac{(a-\lambda)^2}{9}$	$q_2 = \frac{a-\lambda}{4}$	$u_2 = \frac{(a-\lambda)^2}{16}$

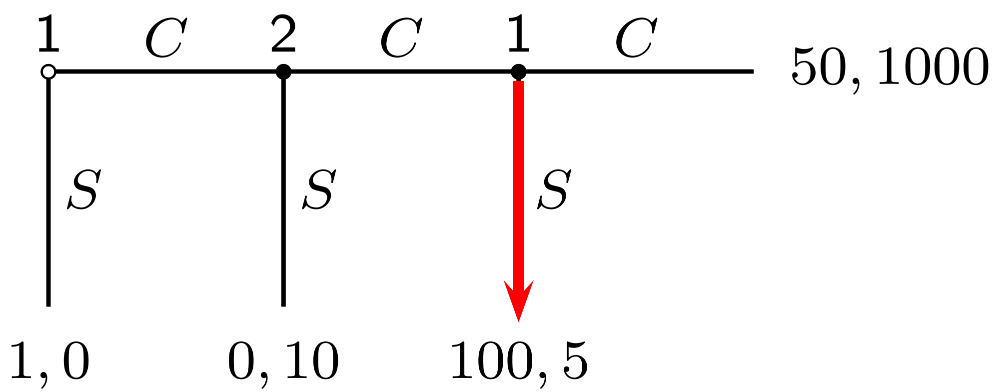
Table 1: Productions and profits in the linear Cournot and Stackelberg duopolies

Backward Induction “Paradox”

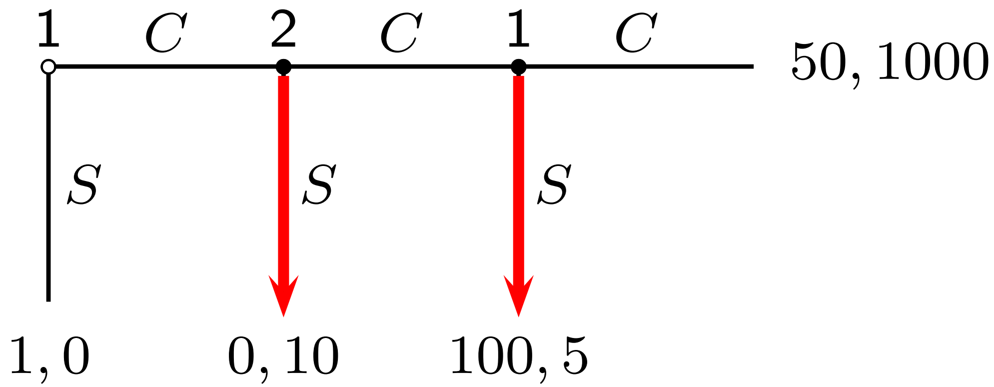
Backward Induction "Paradox"



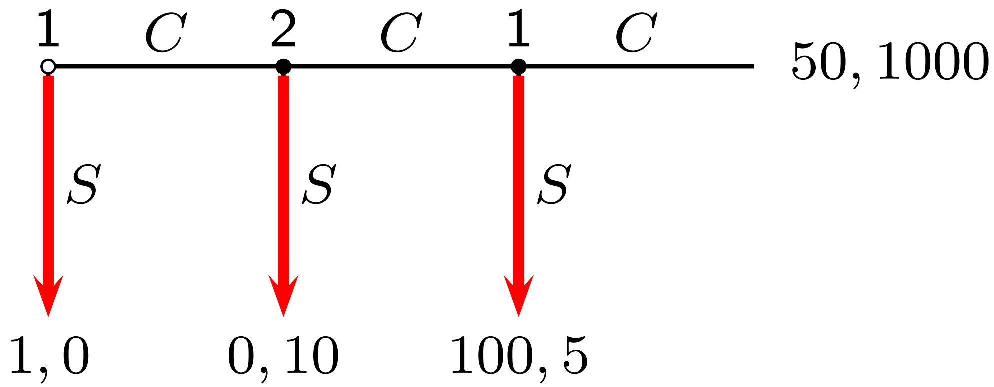
Backward Induction "Paradox"



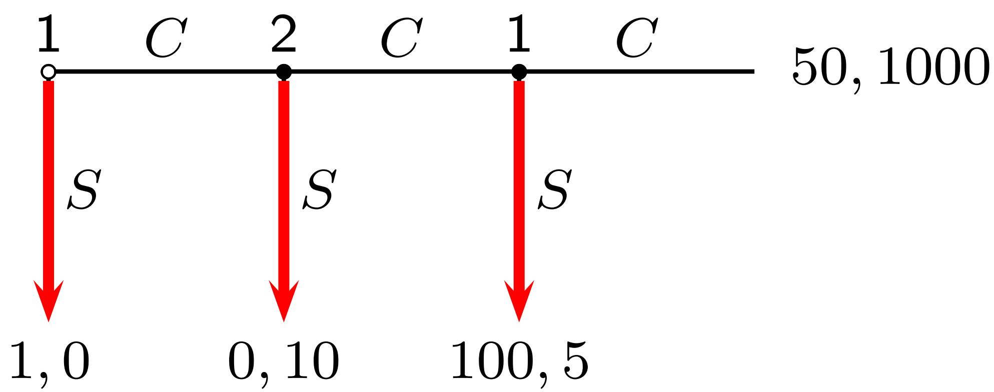
Backward Induction "Paradox"



Backward Induction "Paradox"



Backward Induction "Paradox"



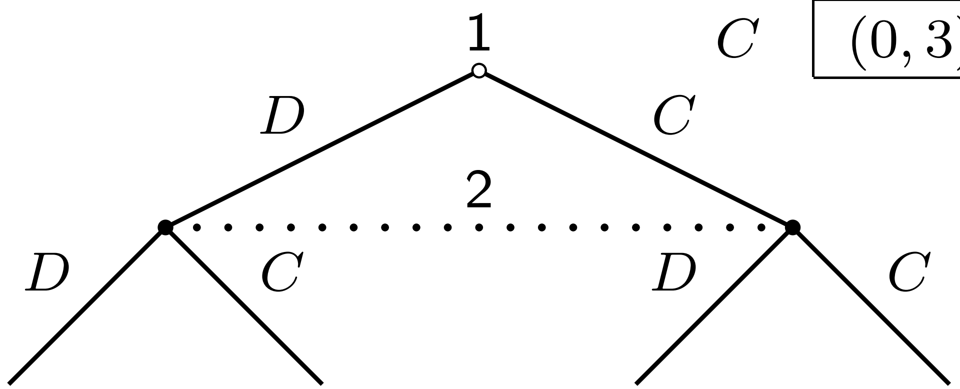
What should player 2 do/think if he actually has to play?

The prisoner dilemma played twice.

	D	C
D	$(1,1)$	$(3,0)$
C	$(0,3)$	$(2,2)$

The prisoner dilemma played twice.

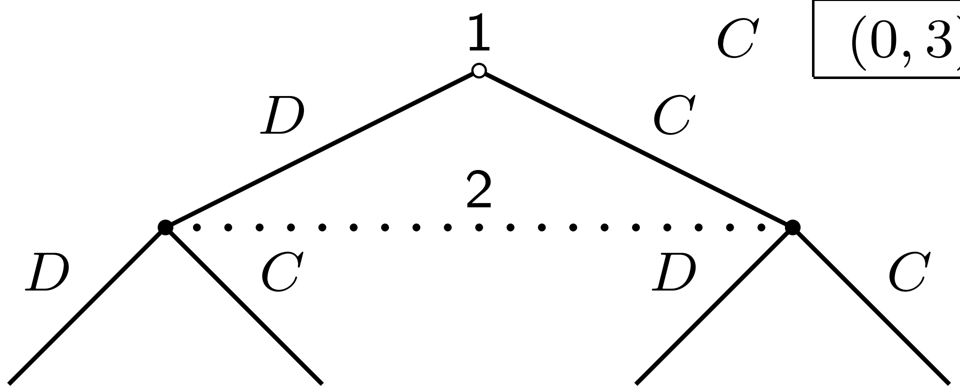
	D	C
D	$(1,1)$	$(3,0)$
C	$(0,3)$	$(2,2)$



	D	C	D	C	D	C	D	C
D	$2,2$	$4,1$	$4,1$	$6,0$	$1,4$	$3,3$	$3,3$	$5,2$
C	$1,4$	$3,3$	$3,3$	$5,2$	$0,6$	$2,5$	$2,5$	$4,4$

The prisoner dilemma played twice.

	D	C
D	$(1,1)$	$(3,0)$
C	$(0,3)$	$(2,2)$

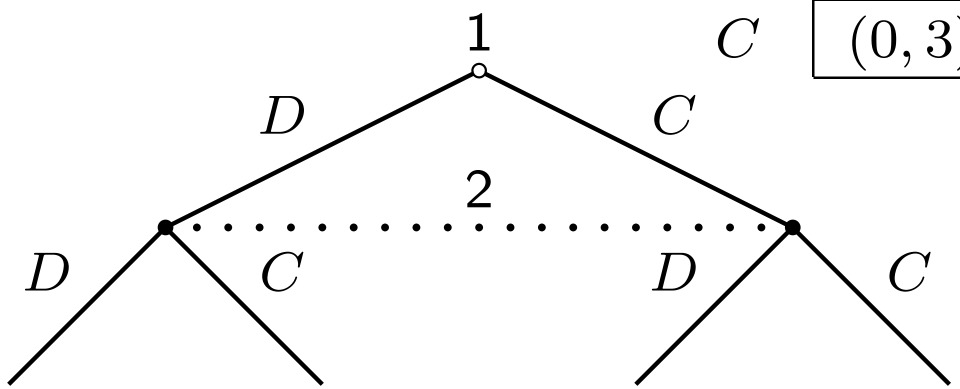


	D	C	D	C	D	C	D	C
D	$2,2$	$4,1$	$4,1$	$6,0$	$1,4$	$3,3$	$3,3$	$5,2$
C	$1,4$	$3,3$	$3,3$	$5,2$	$0,6$	$2,5$	$2,5$	$4,4$

Unique NE (SPNE): both players defect in both periods

The prisoner dilemma played twice.

	D	C
D	$(1, 1)$	$(3, 0)$
C	$(0, 3)$	$(2, 2)$



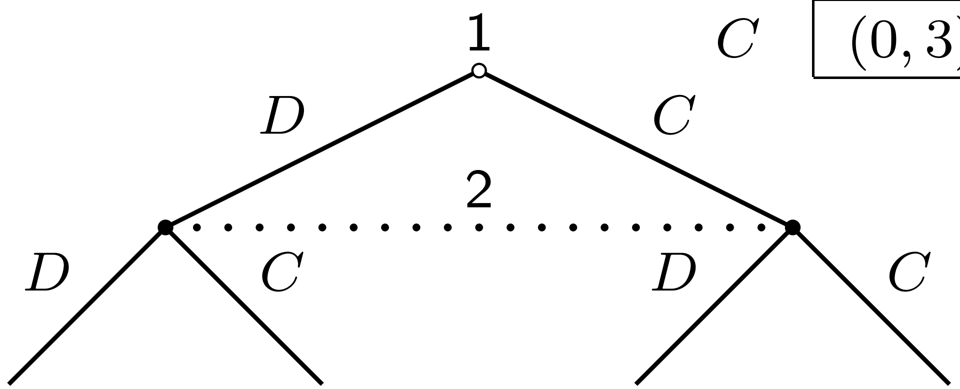
	D	C	D	C	D	C	D	C
D	$2, 2$	$4, 1$	$4, 1$	$6, 0$	$1, 4$	$3, 3$	$3, 3$	$5, 2$
C	$1, 4$	$3, 3$	$3, 3$	$5, 2$	$0, 6$	$2, 5$	$2, 5$	$4, 4$

Unique NE (SPNE): both players defect in both periods

☞ Same result whatever the length (finite and commonly known) of the game

The prisoner dilemma played twice.

	D	C
D	$(1,1)$	$(3,0)$
C	$(0,3)$	$(2,2)$



	D	C	D	C	D	C	D	C
D	$2,2$	$4,1$	$4,1$	$6,0$	$1,4$	$3,3$	$3,3$	$5,2$
C	$1,4$	$3,3$	$3,3$	$5,2$	$0,6$	$2,5$	$2,5$	$4,4$

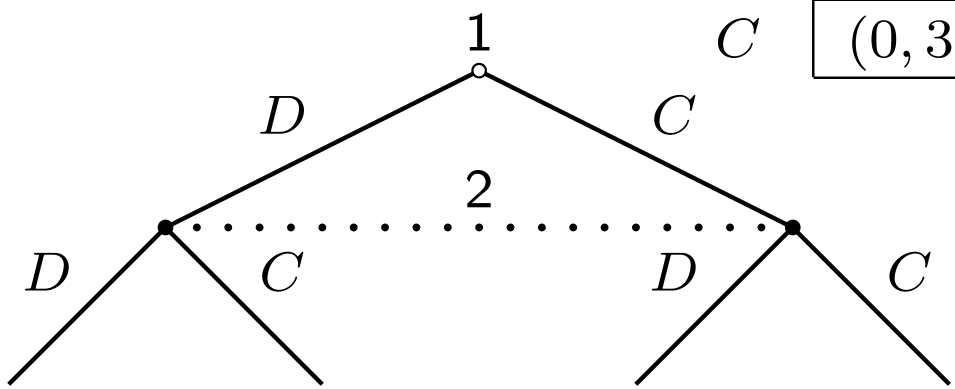
Unique NE (SPNE): both players defect in both periods

☞ Same result whatever the length (finite and commonly known) of the game

What should a player do (think) if his partner cooperate?

The prisoner dilemma played twice.

	D	C
D	$(1,1)$	$(3,0)$
C	$(0,3)$	$(2,2)$



	D	C	D	C	D	C	D	C
D	$2,2$	$4,1$	$4,1$	$6,0$	$1,4$	$3,3$	$3,3$	$5,2$
C	$1,4$	$3,3$	$3,3$	$5,2$	$0,6$	$2,5$	$2,5$	$4,4$

Unique NE (SPNE): both players defect in both periods

☞ Same result whatever the length (finite and commonly known) of the game

What should a player do (think) if his partner cooperate?

Remark. We will see that **infinite** repetition allows cooperation

References

HARSANYI, J. C. (1967–1968): "Games with Incomplete Information Played by Bayesian Players. Parts I, II, III," Management Science, 14, 159–182, 320–334, 486–502.

KUHN, H. W. (1953): "Extensive Games and the Problem of Information," in Contributions to the Theory of Games, ed. by H. W. Kuhn and A. W. Tucker, Princeton: Princeton University Press, vol. 2.

SELTEN, R. (1965): "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit," Zeitschrift für dis gesamte Staatswissenschaft, 121, 301–324 and 667–689.