Repeated Games

Frédéric KOESSLER

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- Definitions: Discounting, Individual Rationality
- Finitely Repeated Games
- Infinitely Repeated Games
- Automaton Representation of Strategies
- The One-Shot Deviation Principle
- Folk Theorems
- Applications: Prisoner Dilemma, Cournot Oligopoly

Main References:

- Mailath and Samuelson (2006): "*Repeated Games and Reputations*"
- Osborne (2004): "*An Introduction to Game Theory*", chap. 14–15
- Osborne and Rubinstein (1994): "*A Course in Game Theory*", chap. 8–9

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- A discount factor may be introduced

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- Average discounted payoff:

$$\frac{\sum_{t=1}^{T} \delta^{t-1}\, x(t)}{\sum_{t=1}^{T} \delta^{t-1}} = \begin{cases} \sum_{t=1}^{T} \frac{x(t)}{T} & \text{if } \delta = 1 \\ x(1) & \text{if } \delta = 0 \end{cases}$$

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  - In each stage, the game stops with probability $(1-\delta)$
  - Players can borrow and lend at the interest rate r

  $\Rightarrow \delta = \frac{1}{1+r}$ $(1 + r$ € tomorrow $\sim \delta(1+r) = 1$ € today)

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$$v_i = \min_{\sigma_{-i} \in \prod_{j \neq i} \Delta(A_j)} \max_{a_i \in A_i} u_i(a_i, \sigma_{-i})$$

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Remark In general, minmax $\neq$ maxmin in a game with more than two players. In 2-player games v_i is also the maximum payoff player 1 can guarantee (maxminimized payoff in mixed strategies)

- A payoff profile $w = (w_1, \ldots, w_n)$ is (strictly) **individually rational** if each player's payoff is larger than his minmax payoff: for every $i \in N$,

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- Outcome / trajectory generated by s:
  $a^1 = s^1,\ a^2 = s^2(a^1),\ a^3 = s^3(a^1, a^2), \ldots$

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Remark If we weaken the equilibrium concept by asking only for approximate best responses (ε-Nash equilibrium) then we can support cooperation for any $\varepsilon > 0$ in the prisoner dilemma if the horizon T is sufficiently large

A Variant of the Prisoner Dilemma.

	D	C	P
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⇒ a Nash equilibrium of a finitely repeated game does not necessarily consist in playing Nash equilibria of the stage game, even if the stage game has a unique Nash equilibrium

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Proposition 2 *If the stage game G has a unique Nash equilibrium then for every finite T and every discount factor $\delta \in (0,1]$, the finitely repeated game $G(T,\delta)$ has a unique SPNE, in which the Nash equilibrium of the stage game is played after all histories*

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But in this example players punish with a "bad" Nash equilibrium. There is therefore an incentive to "Renegotiate" in the second stage if (C,C) is not played in the first stage

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C	$(0,3)$	$(2,2)$	$(0,0)$	$(-2,0)$
M	$(0,-2)$	$(0,-2)$	$(2,-1)$	$(-2,-2)$
N	$(0,0)$	$(0,0)$	$(0,0)$	$(-1,2)$

Three pure strategy Nash equilibria in the stage game:

An example with no incentive to "renegotiate".

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Three pure strategy Nash equilibria in the stage game:

$$(D,D),\ (M,M), \text{ and } (N,N)$$

(not Pareto ordered)

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M	$(0, -2)$	$(0, -2)$	$(2, -1)$	$(-2, -2)$
N	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(-1, 2)$

A SPNE in the 2-stage repeated game (without discounting) with no incentive to renegotiate:

	D	C	M	N
D	$(1,1)$	$(3,0)$	$(0,0)$	$(-2,0)$
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– First stage: $s_i^1 = C$

– Second stage: $s_1^2(a_1^1, a_2^1) = \begin{cases} D & \text{if } (a_1^1, a_2^1) = (C, C) \text{ or } \{a_1^1 \text{ and } a_2^1 \neq C\} \\ M & \text{if } a_1^1 = C \text{ and } a_2^1 \neq C \\ N & \text{if } a_1^1 \neq C \text{ and } a_2^1 = C \end{cases}$

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Exercise 1 ✍ Consider the following stage game.

	A	B	C	D
A	$(4,4)$	$(0,0)$	$(18,0)$	$(1,1)$
B	$(0,0)$	$(6,6)$	$(0,0)$	$(1,1)$
C	$(0,18)$	$(0,0)$	$(13,13)$	$(1,1)$
D	$(1,1)$	$(1,1)$	$(1,1)$	$(0,0)$

(i) Find the pure-strategy NE

(ii) Consider the 2-period repeated game. Find a SPNE with undiscounted average payoff equal to 3 for each player

(iii) To see how to construct equilibria with increasingly severe punishments as the length of the game increases, consider the 3-period repeated game. Find a SPNE with undiscounted average payoff equal to $\frac{13+6+6}{3} = 25/3$ for each player (hint: use the strategy found in (ii) as a punishment for the last two stages)

Infinitely Repeated Games

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Definition Given a normal form game $G = \langle N, (A_i), (u_i) \rangle$, the **infinitely repeated game** $G(\infty, \delta)$ is the extensive form game in which the stage game G is played infinitely often, past actions are publicly observed (perfect monitoring), and players' payoff is the δ-discounted average payoff

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☞ Convex combination, conv$(u(A))$, of all possible payoffs of the stage game

Example. Feasible payoffs in a prisoner dilemma

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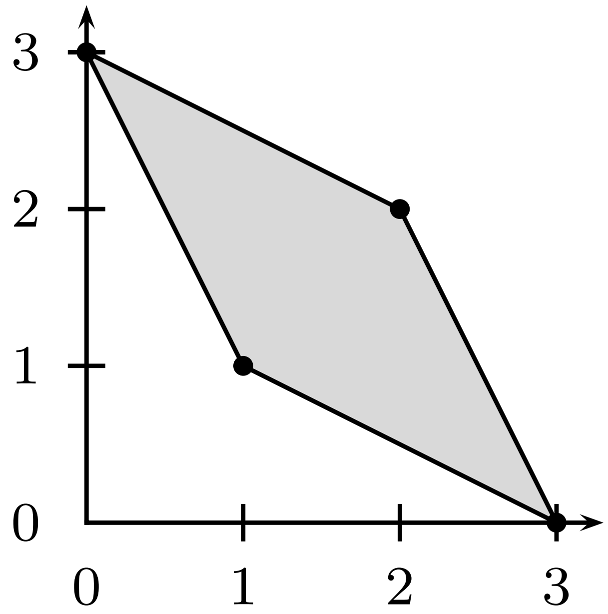
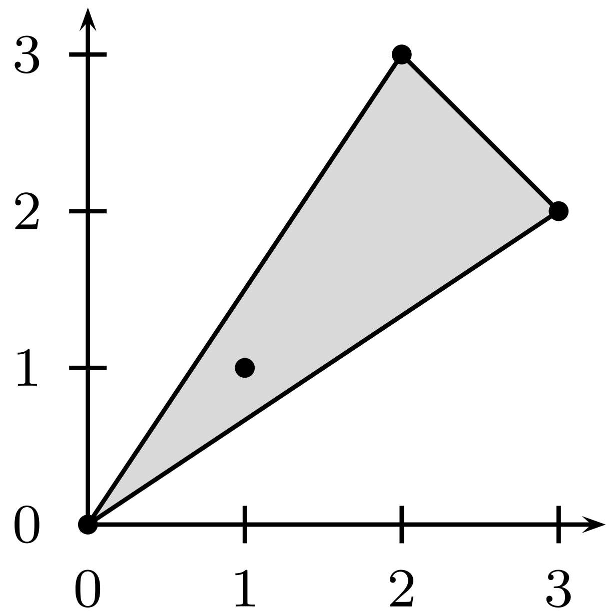


Example. Feasible payoffs in a "battle of sexes" game

	a	b
a	$(3,2)$	$(1,1)$
b	$(0,0)$	$(2,3)$



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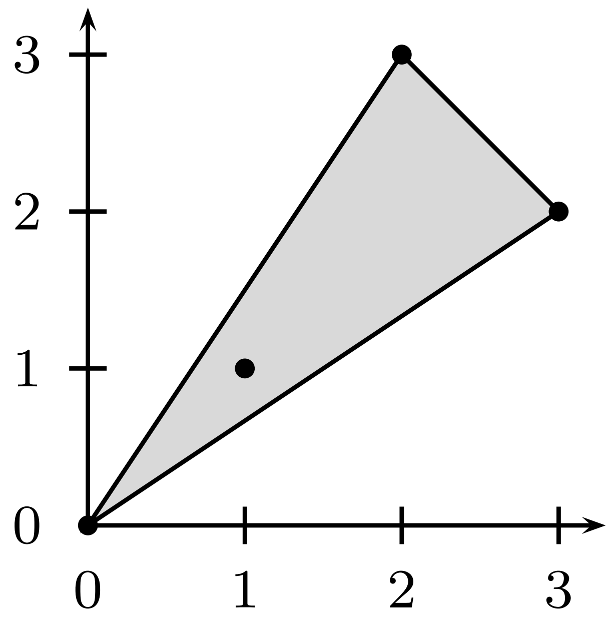

Remark The set of feasible payoffs is usually strictly larger than the set of expected payoffs achievable with mixed (independent) strategies of the one-shot game. For example, the expected payoff profile $(2.5, 2.5)$ is not achievable with mixed strategies in the one-shot battle of sexes

Automaton Representation of Strategies

Automaton for i in the infinitely repeated game:

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- The complexity of a strategy is sometimes defined by the number of states of the smallest automaton that implements it

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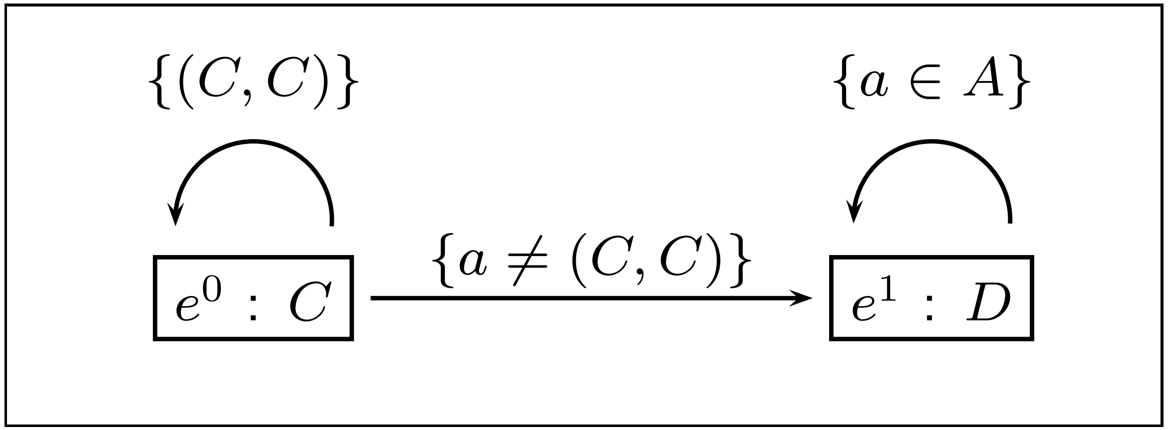

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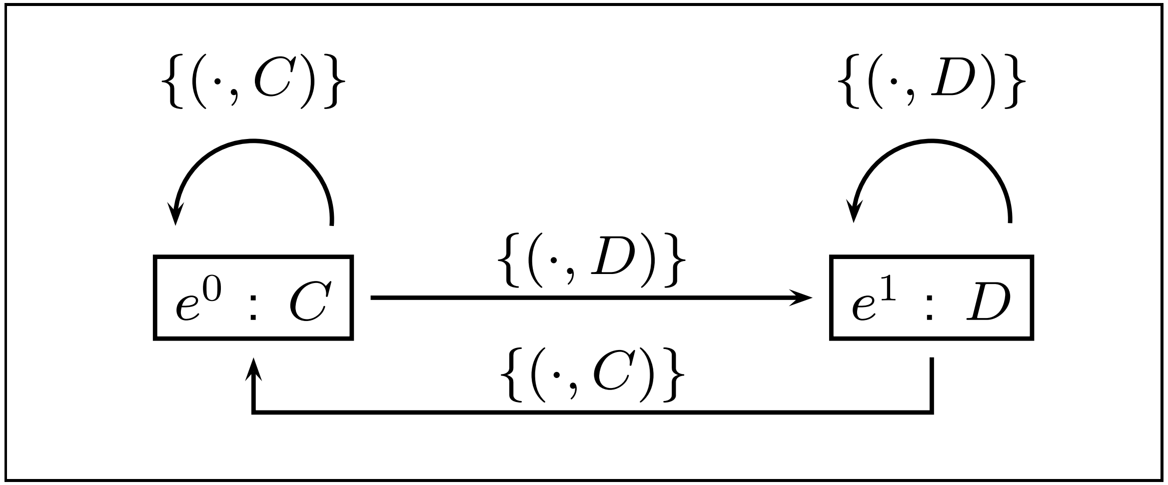

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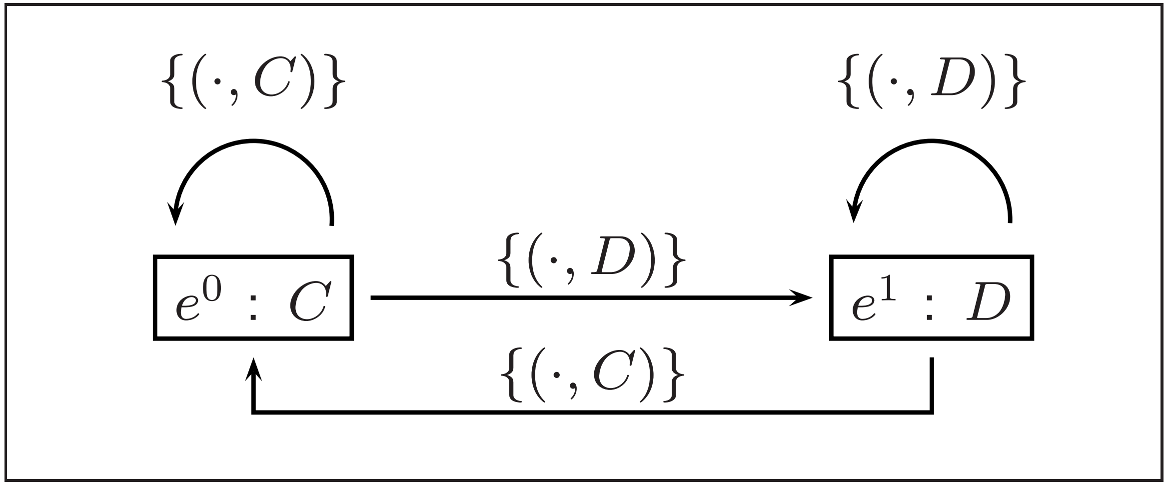

Both players play "*grim*" or "*Tit for Tat*" $\Rightarrow$ cooperation in every period

Exercise 2 ✍ Consider the infinitely repeated PD, with G equal to

	D	C
D	$(1, 1)$	$(3, 0)$
C	$(0, 3)$	$(2, 2)$

(i) Consider the following strategy of player 1: start to cooperate, continue to cooperate as long as player 2 cooperates, and defect for two periods and go back to cooperation if player 2 defects. Write and represent the simplest automaton implementing this strategy

(ii) Consider the following strategy of player 2: cooperate in odd periods and defect in even periods, whatever the actions of player 1. Write and represent the simplest automaton implementing this strategy

(iii) Calculate the undiscounted average payoffs of both players when they play the previous strategy profile

(iv) Find a (pure) strategy that cannot be implemented with a finite automaton

Given a strategy σ_i of player i, let

$$\sigma_i|_{h^t}$$

be the continuation strategy of player i induced by history $h^t \in A^t$, i.e., the strategy implied by σ_i in the continuation game that follows h^t

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Definition A strategy profile σ is a subgame perfect Nash equilibrium of the infinitely repeated game if for all histories h^t, $\sigma|_{h^t}$ is a Nash equilibrium of the repeated game

Definition A one-shot deviation for player i from strategy σ_i is a strategy $\hat{\sigma}_i \neq \sigma_i$ with the property that there exists a unique history $\tilde{h}^t$ such that for all $h^\tau \neq \tilde{h}^t$:

$$\sigma_i(h^\tau) = \hat{\sigma}_i(h^\tau)$$

Hence, a one-shot deviation agrees with the original strategy everywhere except at one history $\tilde{h}^t$ where the one-shot deviation occurs

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Example 2 Consider the Tit for Tat strategy profile in the following PD, leading to an average discounted payoff of 3

	D	C
D	$(1, 1)$	$(4, -1)$
C	$(-1, 4)$	$(3, 3)$

One-shot deviation by player 1 $\Rightarrow$ cyclic outcome DC, CD, DC, CD, ... with average discounted payoff

$$\begin{aligned} & (1-\delta)(4(1+\delta^2+\delta^4+\cdots)-1(\delta+\delta^3+\cdots)) \\ = \quad & (1-\delta)(\frac{4}{1-\delta^2}-\frac{\delta}{1-\delta^2})=\frac{4-\delta}{1+\delta} \end{aligned}$$

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The deviation is not profitable if $\frac{4-\delta}{1+\delta} \leq 3$, i.e., $\delta \geq 1/4$

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But the deviation to perpetual defection (which is **not** a one-shot deviation) is profitable when $(1-\delta)(4+\frac{\delta}{1-\delta}) > 3$, i.e., $\delta < 1/3$

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$\Rightarrow$ For $\delta \in [1/4, 1/3)$ TFT is not a NE despite the absence of profitable one-shot deviations

Exercise 3 ✍ Show that TFT is never a SPNE of the previous infinitely repeated PD whatever the discount factor δ (hint: use the one-shot deviation property in the possible types of subgames)

Conditions for the "*grim*" strategy profile to be a SPNE? We use the OSDP

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$$C \longrightarrow (1-\delta)[V + 3\delta^{t-1} + 3\delta^{t} + 3\delta^{t+1} + \cdots]$$

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Playing D is not a profitable deviation if

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In the subgames off the equilibrium path (i.e., $\exists\, s < t$, a_1^s or $a_2^s = D$) we have

$$C \longrightarrow (1-\delta)[W - 1\delta^{t-1} + 1\delta^{t} + 1\delta^{t+1} + \cdots]$$

$$D \longrightarrow (1-\delta)[W + 1\delta^{t-1} + 1\delta^{t} + 1\delta^{t+1} + \cdots]$$

$\Rightarrow$ a SPNE of an infinitely repeated game does not necessarily consist in playing NE of the stage game in every period, even if the stage game has a unique NE

⇒ a SPNE of an infinitely repeated game does not necessarily consist in playing NE of the stage game in every period, even if the stage game has a unique NE

Exercise 4 ✍ Find the condition on δ for the grim strategy profile to be a SPNE in the prisoner dilemma of Exercise 2

"Folk Theorems"

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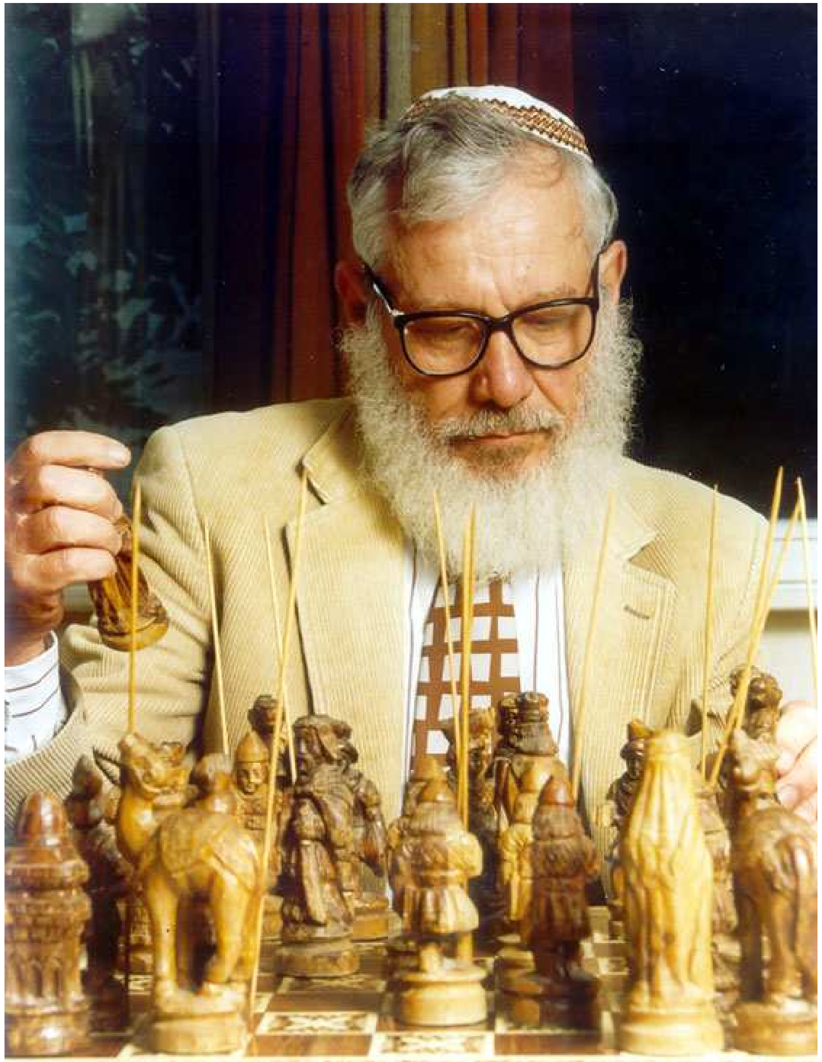

Figure 1: Robert Aumann (1930–), Nobel price in economics in 2005

Proposition 4 *If* $(x_1, \ldots, x_n)$ *is a feasible and strictly individually rational payoff profile, and if* δ *is sufficiently close to 1, then there exits a* **Nash equilibrium** *of the infinitely repeated game* $G(\infty, \delta)$ *in which the discounted average payoff profile is* $(x_1, \ldots, x_n)$

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The folk theorems provide a simple equilibrium characterization. But the negative aspect is that predictive powers are limited

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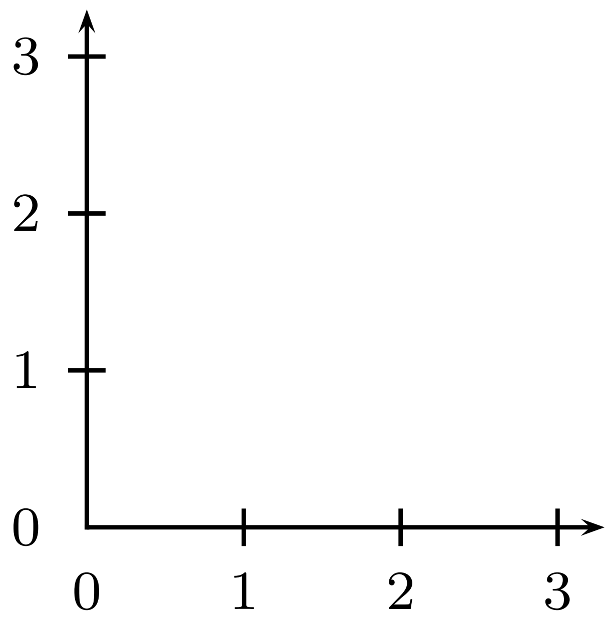

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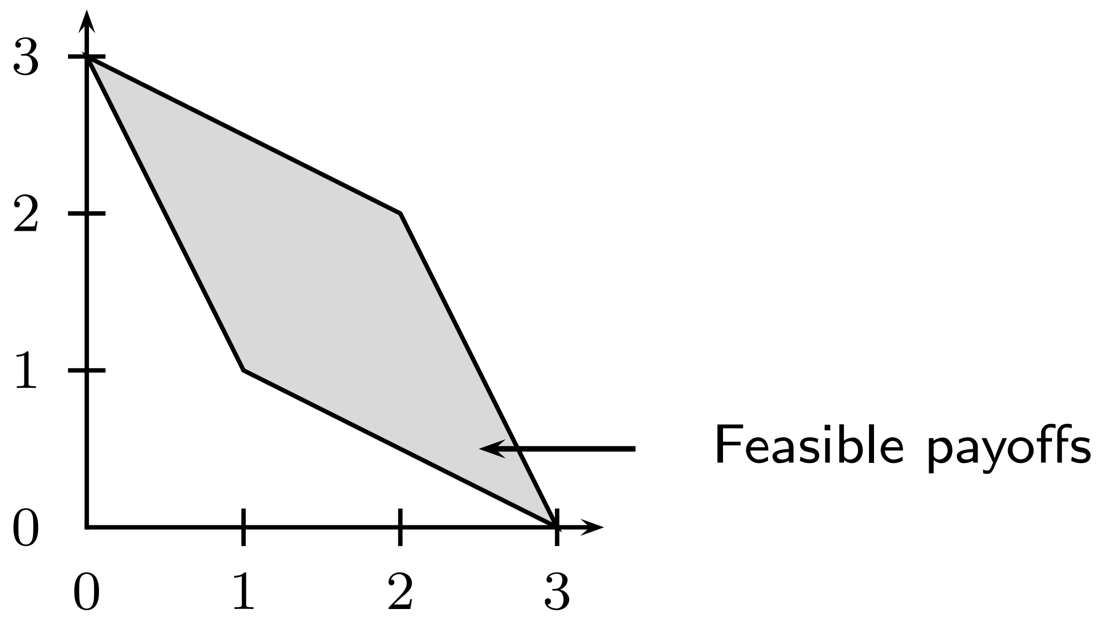

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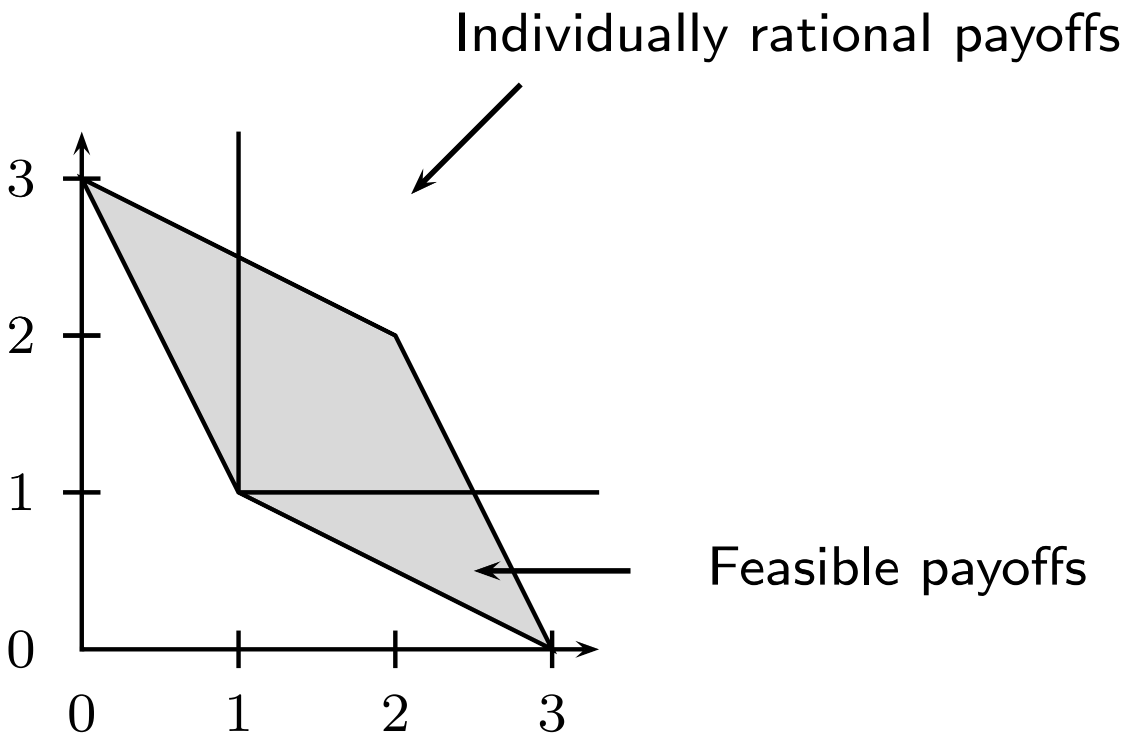

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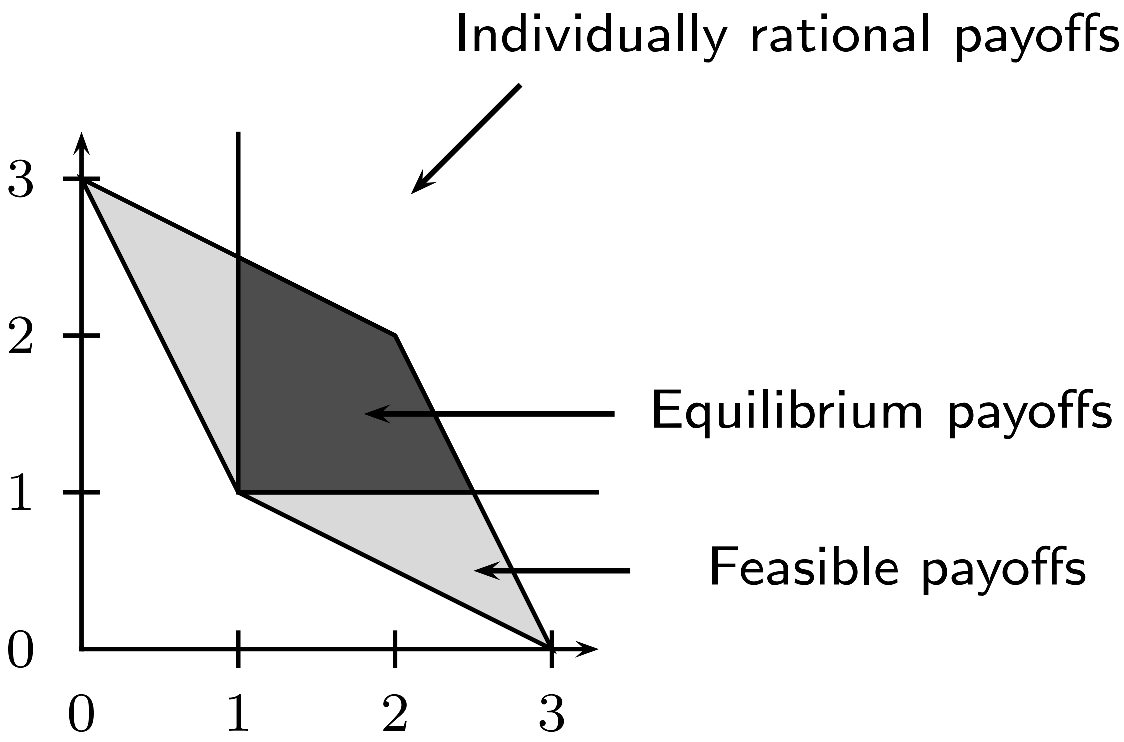

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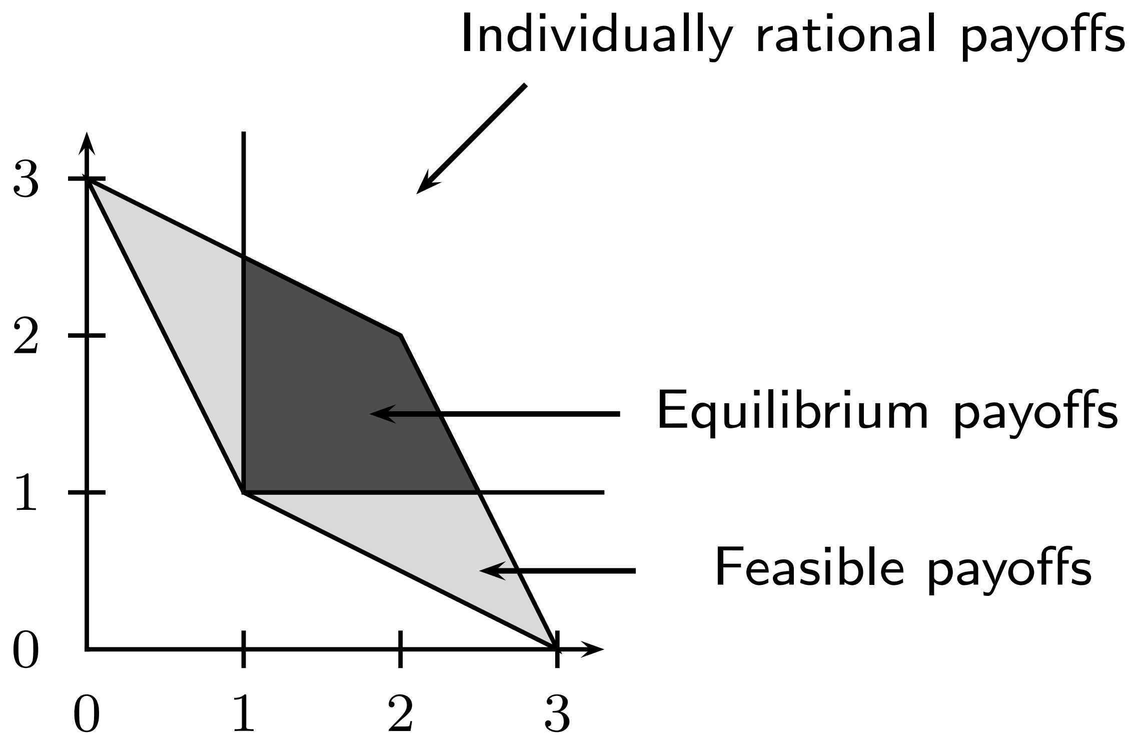

But the prisoner dilemma is special in the sense that the Nash equilibrium payoff profile of the stage game coincides with the minmax payoff profile

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$\Rightarrow$ the equilibrium must be symmetric ($q_i^* = q_i \ \forall\ i$) and $u_i(q_i^*, q_{-i}) = (q_i^*)^2$

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Not necessarily . . .

To simplify, let $c = 0$

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where Y_i is i's profit when i deviates to its stage game best response $BR_i(q_{-i}) = \frac{1-\sum_{j \neq i} q_j}{2} = \frac{1-(n-1)/2n}{2} = \frac{n+1}{4n}$, i.e., $Y_i = (\frac{n+1}{4n})^2$

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$$\frac{1}{4n(1-\delta)} \geq (\frac{n+1}{4n})^2 + \frac{\delta}{(1-\delta)(n+1)^2}$$

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Conclusion: At a SPNE of the infinitely repeated Cournot game the firms can jointly reproduce the monopoly outcome of the market when the discount factor is sufficiently large

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Remark The folk theorem for SPNE is actually more general than in Proposition 5 but use more complicate punishments than Nash equilibria of the stage game. This is irrelevant in the PD because the NE of the stage game is the most severe punishment available. But this last property is not true in all games (e.g., in the Cournot oligopoly game)

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