Cooperative Games

Outline
(September 3, 2007)

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- Introduction
- Nash Bargaining Solution
- Core
- Shapley Value

Introduction

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Basic ingredients of *non-cooperative* games:

- Individuals' strategies
- Outcome of the game = strategy profile
- Players' preferences over outcomes

Basic ingredients of *cooperative* games:

- Actions of *coalitions* (groups of individuals)
- Outcome of the game = formed coalitions ($\rightarrow$ partition of the set of players) and actions of coalitions
- Players' preferences over outcomes (as in non-cooperative games)

Solution concept in cooperative games: set of outcomes for each game

➥ stability (in general), as in non-cooperative games, but towards **groups** of players

Contrary to non-cooperative games, no detail is given on how groups form and make decisions

Negotiation: Cooperative / Axiomatic Approach

➥ Study bargaining games with cooperative game theory

☞ No assumption on how negotiation takes place

☞ Which outcomes have "reasonable" properties?

☞ How does the solution varies with players' preference and opportunities?

➥ Nash bargaining solution

X: set of possible agreements

D: disagreement outcome

$u_i : X \cup \{D\} \to \mathbb{R}$: player i utility function

$\mathcal{U} = \{(v_1, v_2) = (u_1(x), u_2(x)) : x \in X\}$: possible pairs of payoffs

$d = (u_1(D), u_2(D))$: pair of disagreement payoffs

Definition. A *bargaining problem* is a pair $(\mathcal{U}, d)$, where $\mathcal{U}$ is the set of possible payoffs, $d = (d_1, d_2)$ is the disagreement payoff, such that:

(i) $d \in \mathcal{U}$

(ii) There exists $(v_1, v_2) \in \mathcal{U}$ s.t. $v_1 > d_1$ and $v_2 > d_2$

(iii) The set $\mathcal{U}$ is compact (closed and bounded) and convex

5/ **Example.** Exchange economy. Disagreement point $\sim$ initial endowments

Remark. By (ii) the disagreement point d is not Pareto optimal

Definition. A *bargaining solution* is a function ψ that associates with every bargaining problem $(\mathcal{U}, d)$ a unique member $\psi(\mathcal{U}, d)$ of $\mathcal{U}$

Axioms

➥ List of "reasonable" conditions a solution should satisfy

$$\psi(\mathcal{U}, d) = (\psi_1(\mathcal{U}, d), \psi_2(\mathcal{U}, d)) \in \mathcal{U}$$

Remark. Implicit axiom: **existence** and **uniqueness** of $\psi(\mathcal{U}, d)$ for every $(\mathcal{U}, d)$

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✥ **Pareto optimality (PAR).** For every bargaining problem $(\mathcal{U}, d)$, the bargaining solution $\psi(\mathcal{U}, d)$ is not Pareto dominated by a pair (v_1, v_2) of $\mathcal{U}$: $\nexists\ (v_1, v_2) \in \mathcal{U}$ s.t. $v_i \geq \psi_i(\mathcal{U}, d)$, $i = 1, 2$, with at least one strict inequality

➥ No possible renegotiation improving both players' payoffs

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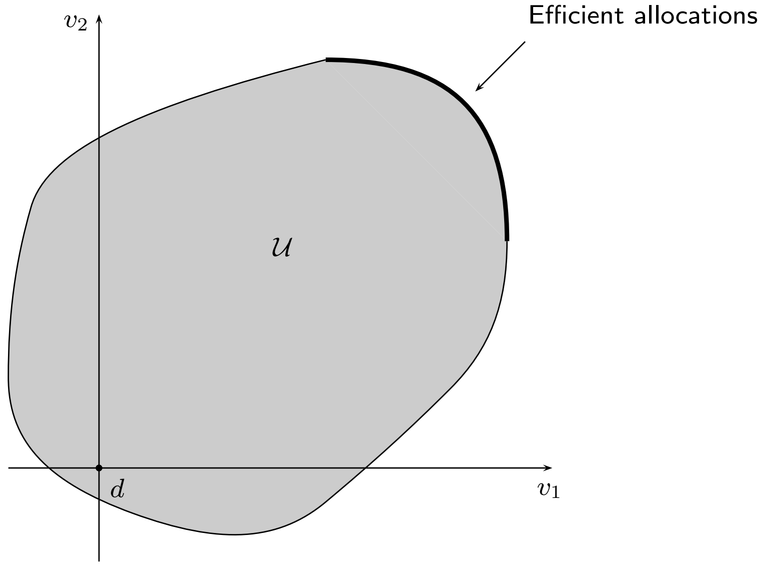

✥ **Symmetry (SYM).** ("Equity") If the bargaining problem $(\mathcal{U}, d)$ is symmetric, i.e., $(v_1, v_2) \in \mathcal{U} \Leftrightarrow (v_2, v_1) \in \mathcal{U}$ (the 45° line is a line of symmetry of $\mathcal{U}$) and $d_1 = d_2$, then the bargaining solution gives every player the same payoff: $\psi_1(\mathcal{U}, d) = \psi_2(\mathcal{U}, d)$

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➥ These two axioms give a unique solution for symmetric games

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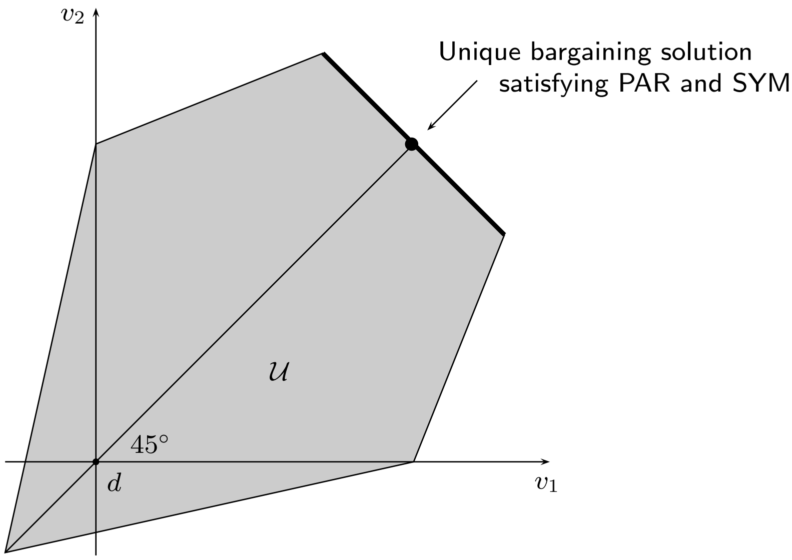

Figure 1:

✥ **Invariance to equivalent payoff representations (INV).** If the bargaining problem $(\mathcal{U}', d')$ is derived from another bargaining problem $(\mathcal{U}, d)$ by an increasing affine transformation ($v_i' = \alpha_i\, v_i + \beta_i$ and $d_i' = \alpha_i\, d_i + \beta_i,\ i = 1,\, 2,\ \alpha_i > 0$), then the solution of the transformed problem for player i is the transformation of the solution of the original problem:

$$\psi_i(\mathcal{U}', d') = \alpha_i\, \psi_i(\mathcal{U}, d) + \beta_i \quad (i = 1,\, 2)$$

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- ➥ Consistency with the cardinality of expected utility functions
- ➥ Without loss of generality we can assume $d = (0, 0)$

$\Rightarrow$ With these three axioms we get a unique solution for every bargaining problem that can be obtained as a linear transformation of a symmetric bargaining problem

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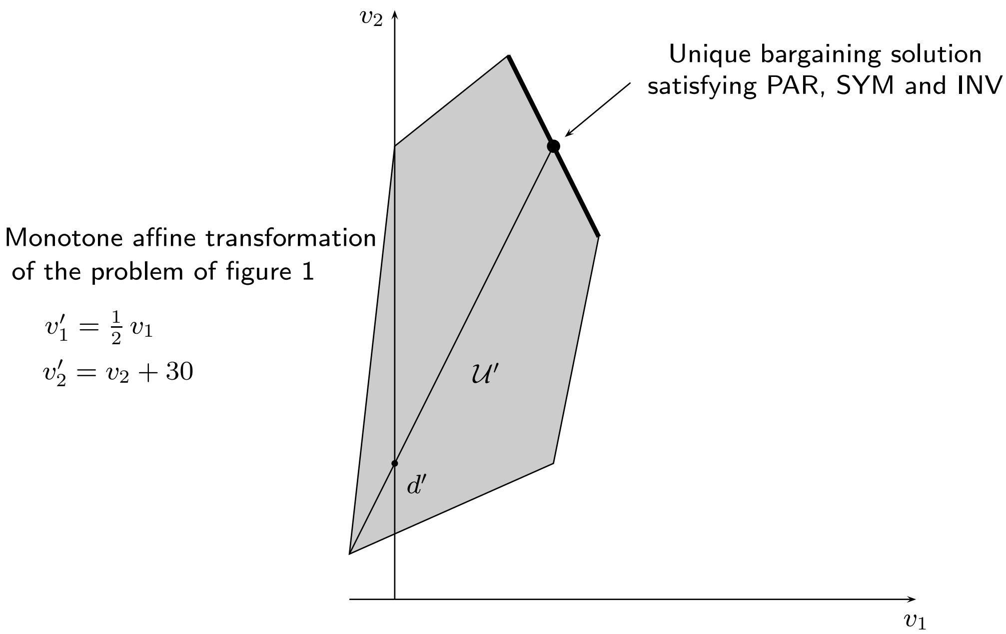

A last axiom is required

❖ **Independence of irrelevant alternatives (IIA).** (invariance to contraction) If two bargaining problems $(\mathcal{U}, d)$ and $(\mathcal{U}', d)$ with the same disagreement point are such that $\mathcal{U} \subseteq \mathcal{U}'$ and $\psi(\mathcal{U}', d) \in \mathcal{U}$ then $\psi(\mathcal{U}, d) = \psi(\mathcal{U}', d)$

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Remark. If ψ is obtained by maximizing a function on $\mathcal{U}$ then this axiom is satisfied

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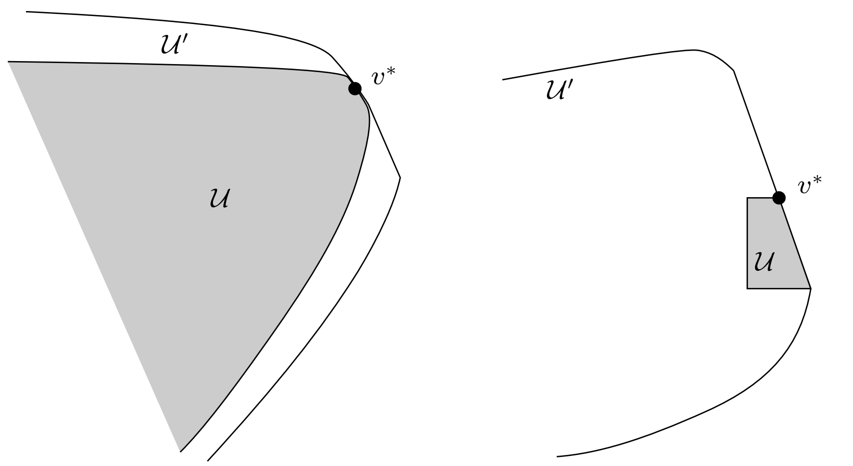


If $\mathcal{U} \subseteq \mathcal{U}'$ and $\psi(\mathcal{U}', d) = v^* \in \mathcal{U}$ then $\psi(\mathcal{U}, d) = v^*$

Proposition. (Nash Theorem) *One and only one bargaining solution satisfies the four axioms PAR, SYM, INV and IIA. It is the* **Nash bargaining solution**, *that assigns to every bargaining problem* $(\mathcal{U}, d)$ *the pair of payoffs that maximizes the* Nash product*:*

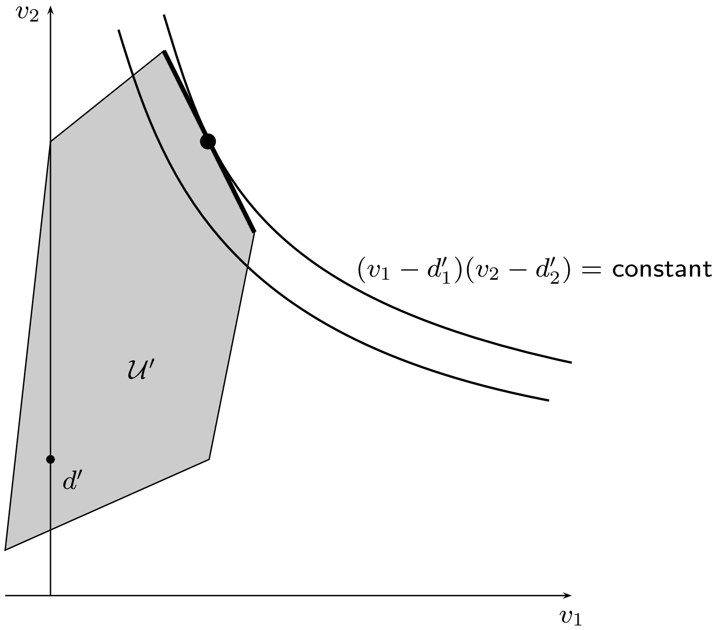
$$\max_v \ (v_1 - d_1)(v_2 - d_2) \quad \textit{s.t.} \quad v \in \mathcal{U} \ \ \textit{and} \ \ v \geq d$$

14/ ✍ Verify that the Nash solution satisfies the 4 axioms ($\Rightarrow$ existence)

For any value of c, the set of points (v_1, v_2) such that

$$(v_1 - d_1)(v_2 - d_2) = c$$

is an hyperbola $\Rightarrow$ the Nash solution is the pair (v_1, v_2) in $\mathcal{U}$ on the highest such hyperbola



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Intuition for the proof of uniqueness.

Let $\psi^N(\mathcal{U}, d) = v^N$ be the Nash solution and $\psi^*(\mathcal{U}, d)$ a solution satisfying the 4 axioms. We show that $\psi^N = \psi^*$

$\boxed{\text{INV}}$ $\Rightarrow$ without loss of generality $d = (0,0)$ and $v^N = (1,1)$ (new scale $\rightarrow \frac{v_i - d_i}{v_i^N - d_i}$)

v^N solves $\max_{v \in \mathcal{U}} (v_1 \cdot v_2) \Rightarrow v^N$ tangent to $v_1 \cdot v_2 = 1$. Equation of the tangent: $v_1 + v_2 = 2$

16/ $\mathcal{U}$ is convex $\Rightarrow \mathcal{U}$ is below the tangent

$\Rightarrow$ we can include $\mathcal{U}$ into a large symmetric rectangle $\mathcal{U}'$ (see figure)

$\boxed{\text{PAR}}$, $\boxed{\text{SYM}}$ $\Rightarrow \psi^*(\mathcal{U}', d) = v^N$

$\boxed{\text{IIA}}$ $\Rightarrow \psi^*(\mathcal{U}, d) = \psi^*(\mathcal{U}', d) = v^N$ because $\mathcal{U} \subseteq \mathcal{U}'$

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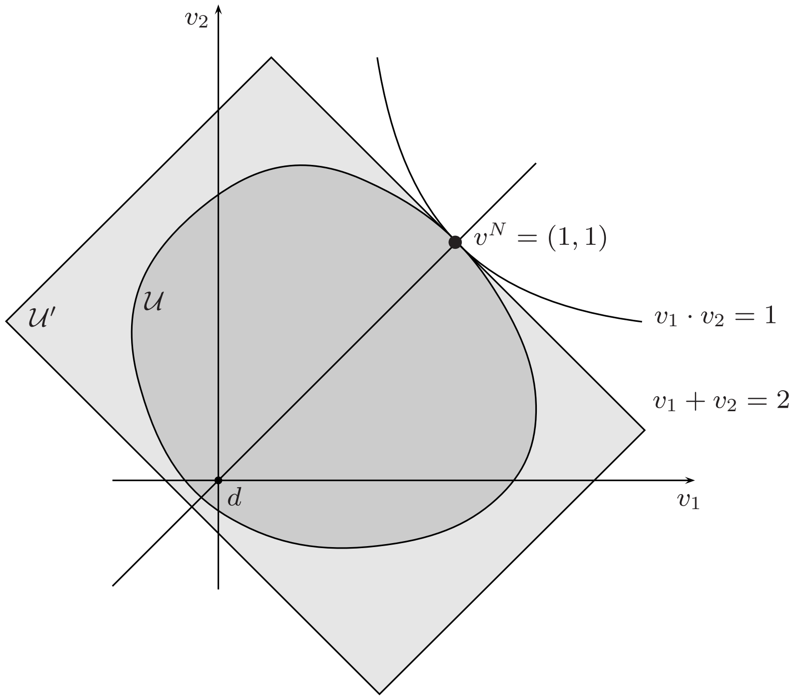

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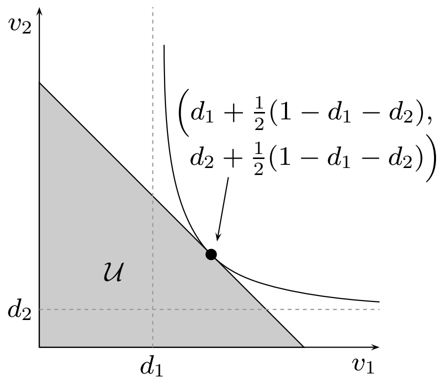
Link with the Strategic Approach

("Nash Program")

Consider a bargaining problem $(\mathcal{U}, d)$ where $\mathcal{U} = \{(v_1, v_2) \in \mathbb{R}^2_+ : v_1 + v_2 \leq 1\}$



☞ The Nash solution is the SPNE outcome of the alternating offers bargaining game with risk of breakdown $\alpha \to 0$ (without discounting), where $d = b$ is the pair of payoffs when negotiations terminate (Binmore et al., 1986)

Generalization to n players?

- **1st obvious solution**: $\mathcal{U} \subseteq \mathbb{R}^n$, disagreement point $d = (d_1, \ldots, d_n) \in \mathcal{U}$

Interpretation: either all agree on $v \in \mathcal{U}$, or disagreement d

$$\Rightarrow \quad \max_{v \in \mathcal{U}} \prod_{i=1}^{n} (v_i - d_i) \quad \text{s.t.} \quad v \geq d$$

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... but this solution ignores *coalitions* formations and their influences on the solution

- **2nd solution**: taking into coalitions formations, or at least the potential threat of coalitions formations

Coalitions and Characteristic Functions

(September 3, 2007)

Coalitional game: model of interactive decisions based on the behavior of coalitions of players

A coalition is a subset of players $S \subseteq N \equiv \{1, \ldots, n\}$, $S \neq \emptyset$ ($2^n - 1$ possible coalitions)

$S = \{i\}$: coalition of one player (singleton)

20/ $S = N$: coalition of all players (grand coalition)

Assumption: *Transferable Utility games*: we can make the sum of players' utilities in a coalition and redistribute it to its members

Definition. A *TU coalitional game*, or *game in characteristic form*, is a pair (N, v) where

- N is the set of players
- v is a *characteristic function* which associates a *value* $v(S) \in \mathbb{R}$ to each coalition S of N

For every coalition S, $v(S)$ is the total payoff for members of coalition S (independently of players' behavior outside S)

➥ $v(S)$ = a priori power of group S

Definition. A game is

- *symmetric* if the value of a coalition only depends on its size: there is a function f such that $v(S) = f(|S|)$ for all $S \subseteq N$
- *monotonic* if $S \subseteq T \Rightarrow v(S) \leq v(T)$

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Assumption: *Superadditivity*: $S \cap T = \emptyset \Rightarrow v(S \cup T) \geq v(S) + v(T)$

Remark.

- Superadditivity $\Rightarrow v(N) \geq \sum_k v(S_k)$ for every partition $\{S_k\}_k$ of N
- If $v(S) \geq 0 \ \forall\ S$ then superadditivity implies monotonicity

✍ Find a superadditive game which is not monotonic

Simple Games

A coalitional game (N, v) is **simple** if $v(S) = 1$ (*winning coalition*) or $v(S) = 0$ (*loosing coalition*), and $v(N) = 1$

Remark. By superadditivity, if $v(S) = 1$ then $v(N \backslash S) = 0$ and $v(T) = 1$ for $S \subseteq T$ (but not $\Leftarrow$)

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A player j has a *veto power* if he belongs to all winning coalitions ($v(S) = 1 \Rightarrow j \in S$)

A player j is a *dictator* if a coalition is winning iff player j belongs to it ($v(S) = 1 \Leftrightarrow j \in S$)

Examples. (3 players)

- *Simple majority*. A coalition is winning iff it includes at least 2 members

  ➥ $\begin{cases} v(1) = v(2) = v(3) = 0 \\ v(1,2) = v(1,3) = v(2,3) = v(1,2,3) = 1 \end{cases}$

- *Unanimity*. Only the grand coalition is winning

  ➥ $\begin{cases} v(1,2,3) = 1 \\ v(S) = 0 \text{ for the other coalitions} \end{cases}$

23/
- *Veto game*. A coalition is winning iff it includes player 2 and at least one other player

  ➥ $\begin{cases} v(1) = v(2) = v(3) = v(1,3) = 0 \\ v(1,2) = v(2,3) = v(1,2,3) = 1 \end{cases}$

- *Dictatorship*. A coalition is winning iff it includes player 2

  ➥ $\begin{cases} v(1) = v(3) = v(1,3) = 0 \\ v(2) = v(1,2) = v(2,3) = v(1,2,3) = 1 \end{cases}$

Problem: how to share $v(N)$ among the n players?

The Core

No coalition can increase the payoff of all its members by deviating

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For any payoff profile $(x_i)_{i\in N}$ and coalition S we denote by $x(S) = \sum_{i\in S} x_i$ the sum of payoffs of members in S

Definition. A payoff profile $(x_i)_{i\in N}$ is S-*feasible* if $x(S) = v(S)$. It is *feasible* if it is N-feasible

Definition. The **core** of a coalitional game (N, v) is the set of feasible allocations $(x_i)_{i \in N}$ such that

$$x(S) \geq v(S) \quad \forall\, S \subseteq N$$

or, equivalently, such that there is no coalition S and S-feasible allocation $(y_i)_{i \in N}$ with $y_i > x_i$ for every $i \in S$

25/ ☞ The allocation $(x_i)_{i \in N}$ cannot be *blocked* by a coalition S ("social stability")

Remark. Collective rationality ($x(N) = v(N)$) and individual rationality ($x_i \geq v(i)\ \forall\, i$) are satisfied

Examples

Simple Games.

Majority.

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$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_i \geq 0, \quad \forall\, i \\ x_1 + x_2 \geq 1 \\ x_1 + x_3 \geq 1 \\ x_2 + x_3 \geq 1 \end{cases} \quad \Rightarrow \quad \text{impossible (core} = \emptyset)$$

Unanimity.

$$\text{Core} = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1,\ x_i \geq 0\ \forall\, i\}$$

Veto power.

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_i \geq 0, \quad \forall\, i \\ x_1 + x_2 \geq 1 \\ x_2 + x_3 \geq 1 \end{cases} \quad \Rightarrow \text{ Core} = \{(0,1,0)\}$$

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Dictatorship.

$$\text{Core} = \{(0,1,0)\}$$

➥ No difference between veto and dictatorship. (The Shapley value will make a difference)

Proposition. *In a simple game,*

(i) if no player has a veto power then the core is empty

(ii) if at least one player has a veto power the core is non-empty: it is the set of positive and feasible allocations giving zero payoff to all non-veto players

Proof.

28/ (i) No player has a veto power $\Leftrightarrow \forall\, i \in N$, $\exists\, S$ s.t. $v(S) = 1$ and $i \notin S$, so $v(N \backslash i) = 1$ for all i (monotonicity)

$x \in$ Core $\Rightarrow x(N) = 1$ and $x(N \backslash i) \geq v(N \backslash i) = 1$ for all $i \Rightarrow$ impossible

(ii) Let $V \neq \emptyset$ be the set of veto players and x a positive and feasible allocation giving zero payoff to all non-veto players:

$$\left.\begin{array}{l} x_i \geq 0 \quad \forall\, i \in V \\ x_i = 0 \quad \forall\, i \notin V \\ \displaystyle\sum_{i \in N} x_i = 1 \end{array}\right\} \tag{1}$$

- If S is winning then $V \subseteq S$, so $x(S) = 1 = v(S)$
- If S is loosing then $v(S) = 0$, so $x(S) \geq v(S)$

thus $x \in$ core

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To show that only allocations (1) belong to the core, let x be a core allocation that does not satisfy (1), i.e., $x_j > 0$ for one $j \notin V$

$j \notin V \overset{\text{def}}{\Rightarrow} \exists\, S,\, j \notin S$, s.t. $v(S) = 1 > x(S)$, so S blocks x, i.e. $x \notin$ core □

General necessary and sufficient conditions for the core to be non-empty: Bondareva (1963) and Shapley (1967) (see Osborne and Rubinstein, 1994, pp. 262–263)

A Production Economy

Firm (landowner): player 0

K workers: players $1, \ldots, K$

k workers with the landowner can produce $f(k) \geq 0$, where $f \nearrow$, concave and $f(0) = 0$. Without the landowner they produce nothing

$$\Rightarrow \quad \begin{cases} N = \{0, 1, \ldots, K\} \\ v(S) = \begin{cases} 0 & \text{if } 0 \notin S \\ f(|S| - 1) & \text{if } 0 \in S \end{cases} \end{cases}$$

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Core :

$$x_0 + x_1 + \cdots + x_K = f(K) \tag{2}$$

$$x_i \geq 0, \quad \forall\, i \tag{3}$$

$$x(S) \geq f(|S| - 1) \quad \text{if } 0 \in S \tag{4}$$

(4) $\Rightarrow x(N \backslash i) \geq f(K-1)\ \forall\, i \neq 0 \overset{(2)}{\Rightarrow} f(K) - x_i \geq f(K-1) \Rightarrow$
$x_i \leq f(K) - f(K-1)\ \forall\, i \neq 0$

We showed that $x \in$ core $\Rightarrow x$ belongs to the set

$$x_0 + x_1 + \cdots + x_K = f(K) \tag{2}$$
$$x_i \geq 0, \quad \forall\, i \tag{3}$$
$$x_i \leq f(K) - f(K-1), \quad i = 1, \ldots, K \tag{5}$$

Let us show the converse: let x be in this set

If $0 \notin S$ then $v(S) = 0$ so $x(S) \geq v(S)$

31/ If $0 \in S$ then $x_i \leq f(K) - f(K-1)$ $\forall\, i \in N \backslash S \Rightarrow$
$x(N \backslash S) \leq (K-k)(f(K) - f(K-1))$, where $k = |S| - 1 =$ nb of workers in S

$\Rightarrow x(S) \geq f(K) - (K-k)(f(K) - f(K-1)) \overset{\text{concavity}}{\geq} f(k) = v(S)$

Conclusion: Each worker obtains at best his marginal productivity when all workers are employed, and the landowner gets the remaining payoff

Unionized Workers

- ➥ Only the group of K workers can accept to work

  ➟ $v(S) = \begin{cases} f(K) & \text{if } S = N \\ 0 & \text{otherwise} \end{cases}$

32/ $\Rightarrow$ core $= \{(x_0, x_1, \ldots, x_K) : x_i \geq 0 \;\forall\, i,\; \sum x_i = f(K)\}$

Main Defaults of the Core Solution Concept

① Often too large

② Often empty

③ Extreme and non-robust predictions

- Ex: No difference between veto power and dictator
- Ex: Shoes game

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Shoes Game.

2 players, $i = 1, 2$, each have a left shoe

1 player, $i = 3$, has a right shoe

$v(S) = 1$ € for each pairs of shoes that coalition S can obtain

Core:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_i \geq 0, \quad \forall\, i \\ x_1 + x_3 \geq 1 \\ x_2 + x_3 \geq 1 \end{cases} \quad \Rightarrow \text{ Core} = \{(0, 0, 1)\}$$

Similarly, if

34/ 1 000 001 players have a left shoe

1 000 000 players have a right shoe

the unique core allocation gives 1 € to each owner of a right shoe, and nothing to owners of a left shoe

➠ Relative scarcity of right shoes $\Rightarrow$ price $= 0$ for left shoes (competitive effect)

The Shapley value gives slightly more than 0.5 for right shoes and slightly less than 0.5 for left shoes

Shapley Value

(September 3, 2007)

Classical solution concept for n-person cooperative games with transferable utility (TU games)

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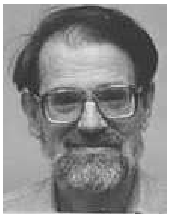

Figure 2: Lloyd Shapley (1923–)

Like the Nash bargaining solution, the Shapley (1953) value is a solution concept satisfying some reasonable axioms (+ existence and uniqueness)

Appropriate solution concept for problems of cost sharing or allocation of resources (telecommunications, joint ownership, . . .)

Characteristic function

$$\begin{aligned} v : 2^N \backslash \emptyset &\rightarrow \mathbb{R}_+ \\ S &\mapsto v(S) \end{aligned}$$

36/ We are looking for a solution

$$\varphi(v) = (\varphi_i(v))_{i \in N}$$

$\varphi_i(v)$ is a *power index* for player i / a value of the game for player i

Axioms

❖ **Axiom 1. Pareto optimality (PAR).**

$$\sum_{i=1}^{n} \varphi_i(v) = v(N)$$

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❖ **Axiom 2. Symmetry (SYM).** If i and j are symmetric (substitutes), i.e.,

$$v(S \cup \{i\}) = v(S \cup \{j\}) \quad \forall\, S \not\ni i, j$$

$$\text{then } \ \varphi_i(v) = \varphi_j(v)$$

❖ **Axiom 3. Null player (NUL).** If i is null, i.e.,

$$v(S \cup \{i\}) = v(S) \quad \forall\, S \not\ni i$$

$$\text{then } \ \varphi_i(v) = 0$$

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❖ **Axiom 4. Linearity (LIN).** Define $(v + w)(S) = v(S) + w(S)$. Then,

$$\varphi(v + w) = \varphi(v) + \varphi(w)$$

(mathematical simplification, but no clear interpretation)

Shapley Theorem. There exists one and only one solution φ satisfying the four preceding axioms. It can be calculated explicitly:

$$\varphi_i(v) = \frac{1}{n\,!} \sum_R \left[v(S_i^R \cup \{i\}) - v(S_i^R)\right]$$

where the sum (R) is over all $n\,!$ permutations of N

and $S_i^R \subseteq N$ is the coalition of players preceding i in order R $\;(v(\emptyset) = 0)$

39/ ➥ $\varphi_i(v)$ is a weighted sum of the marginal contributions of player i

Examples. (3 players)

- *Simple majority / unanimity*

PAR + SYM $\Rightarrow \varphi_1(v) = \varphi_2(v) = \varphi_3(v) = 1/3$

- *Dictator* (player 2)

PAR + NUL $\Rightarrow \varphi_1(v) = \varphi_3(v) = 0$ and $\varphi_2(v) = 1$

- *Veto power* (of player 2)

PAR + SYM $\Rightarrow \varphi_1(v) = \varphi_3(v) = [1 - \varphi_2(v)]/2$

We use the formula to calculate $\varphi_2(v)$:

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$3\,! = 6$ possible orders	Marginal contributions of player 2
123	$v(12) - v(1) = 1$
132	$v(132) - v(13) = 1$
213	$v(2) - v(\emptyset) = 0$
231	$v(2) - v(\emptyset) = 0$
312	$v(312) - v(31) = 1$
321	$v(32) - v(3) = 1$

$$\Rightarrow \;\varphi_2(v) = 4/6 = 2/3$$

$$\Rightarrow \;\varphi(v) = (1/6, 2/3, 1/6)$$

Proposition. *If the game is superadditive then the Shapley value satisfies individual rationality:*

$$\varphi_i(v) \geq v(i) \quad \forall\, i \in N$$

Proof. Superadditivity $\Rightarrow v(S_i^R \cup \{i\}) \geq v(S_i^R) + v(i) \Rightarrow$
$v(S_i^R \cup \{i\}) - v(S_i^R) \geq v(i) \Rightarrow \varphi_i(v) = \frac{1}{n!}\sum_R [v(S_i^R \cup \{i\}) - v(S_i^R)] \geq v(i)$ □

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Shapley value in simple games

Simple games: $v(S) = 0$ or 1 for every S $+$ Monotonicity ($T \subseteq S \Rightarrow v(T) \leq v(S)$)

Player i is *pivotal* in order R if $v(S_i^R) = 0$ and $v(S_i^R \cup \{i\}) = 1$

➡ $$\varphi_i(v) = \frac{\text{nb of orders in which } i \text{ is pivotal}}{n!}$$

Electoral games and political power

Weighted Game :

We assign a *weight* $q_i \geq 0$ to each player i

Quota Q, where $\sum_{i \in N} q_i \geq Q > \sum_{i \in N} q_i/2$

Coalition S is winning ($v(S) = 1$) iff $\sum_{i \in S} q_i \geq Q$

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Examples

- **1 large party and 3 small parties.**

Large party: 1/3 of the electorate $q_1 = 1/3$

Small party: 2/9 of the electorate $q_2 = q_3 = q_4 = 2/9$

Quota $Q = 1/2$ (simple majority)

Minimal winning coalitions: 1 large + 1 small or 3 small

4 equally likely positions for the large party

Pivotal positions: 2nd and 3rd $\Rightarrow$ $\varphi_1(v) = 1/2 > q_1 = 1/3$

$$\Rightarrow \quad \varphi(v) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$$

43/ • **2 large parties and 3 small parties.**

Large party: 1/3 of the electorate $\quad q_1 = q_2 = 1/3$

Small party: 1/9 of the electorate $\quad q_3 = q_4 = q_5 = 1/9$

Minimal winning coalitions: 1 large + 2 small or 2 large

4 equally likely order configurations for a large party, with 5 equally likely positions in each

	●		•		•		•	
N		$\overset{\uparrow}{P}$		$\overset{\uparrow}{P}$		N		N
	•		●		•		•	
N		N		$\overset{\uparrow}{P}$		N		N
	•		•		●		•	
N		N		$\overset{\uparrow}{P}$		N		N
	•		•		•		●	
N		N		$\overset{\uparrow}{P}$		$\overset{\uparrow}{P}$		N

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$$\Rightarrow \quad \varphi_1(v) = \varphi_2(v) = 6/20 = 3/10 < q_1 = q_2 = 1/3$$

$$\Rightarrow \quad \varphi(v) = \left(\frac{3}{10}, \frac{3}{10}, \frac{4}{30}, \frac{4}{30}, \frac{4}{30}\right)$$

- **2 large parties and n small parties, $n \to \infty$.**

4 equally likely order configurations:

❶ 1 and 2 are both in the first half of the ordering

❷ 1 and 2 are both in the second half of the ordering

❸ 1 is in the first half and 2 is in the second half

❹ 2 is in the first half and 1 is in the second half

45/ 1 is pivotal in configuration ❶ if he is after 2, and in configuration ❷ if he is before 2, so he is pivotal in $1/8 + 1/8 = 1/4$ of the situations

$$\Rightarrow \quad \varphi(v) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2n}, \frac{1}{2n}, \cdots\right)$$

Do small parties have an interest to unite?

No, because the game would be symmetric $\Rightarrow$ small parties would share $1/3$ instead of $1/2$

Paradox of the new members of the European union council

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	1958		1973	
Members	Weight	Shapley Val.	Weight	Shapley Val.
France	4	0.233	10	0.179
Germany	4	0.233	10	0.179
Italy	4	0.233	10	0.179
Belgium	2	0.150	5	0.081
Netherlands	2	0.150	5	0.081
Luxembourg	1	0.000	2	0.010
Denmark	–	–	3	0.057
Ireland	–	–	3	0.057
United Kingdom	–	–	10	0.179
Quota	12 over 17		41 over 58	

Luxembourg: null player in 1958. In 1973, relative weight $\searrow$ but power $\nearrow$

Cost Allocation

Value of a visit of H for A, B and C: 20 each. How to share transportation costs of H between A, B and C?

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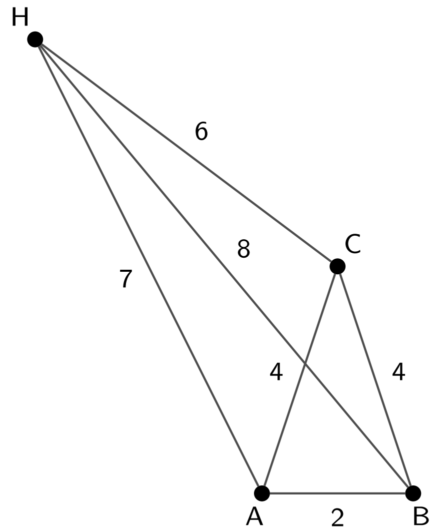

$v(A) = 20 - 14 = 6$
$v(B) = 4$
$v(C) = 8$

$v(AB) = 23$
$v(AC) = 23$
$v(BC) = 22$
$v(ABC) = 60 - 19 = 41$

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	Marginal Contributions		
Possible orders	A	B	C
ABC	6	17	18
ACB	6	18	17
BAC	19	4	18
BCA	19	4	18
CAB	15	18	8
CBA	19	14	8
$\sum_R$	84	75	87
$\varphi = \sum_R / n!$	14	12.5	14.5
Cost allocation	6	7.5	5.5

$v(A) = 6$
$v(B) = 4$
$v(C) = 8$

$v(AB) = 23$
$v(AC) = 23$
$v(BC) = 22$
$v(ABC) = 41$

Other Power Indexes

Banzhaf Index.

Player i is a *key player* in coalition $S \ni i$ if $v(S \backslash \{i\}) = 0$ and $v(S) = 1$

$s_i(v)$: number of coalitions $S \subseteq N$ in which i is a key player

$$➡ \quad \beta_i(v) = \frac{s_i(v)}{\sum_{i \in N} s_i(v)}$$

49/ ➥ Relative number of coalitions in which i is a key player

Example. (Veto power of player 2) $(q_1, q_2, q_3) = (1, 2, 1) \quad Q = 3$

Winning coalitions (key players underlined): $\underline{1}\,\underline{2} \quad \underline{2}\,\underline{3} \quad 1\,\underline{2}\,3$

$$\Rightarrow \quad s_1 = s_3 = 1, \; s_2 = 3, \; \sum_i s_i = 5$$

$$\Rightarrow \quad \beta = \left(\frac{1}{5}, \frac{3}{5}, \frac{1}{5}\right) \neq \varphi = \left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$$

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