Cooperative Games

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Outline

(September 3, 2007)

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- Introduction

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- Introduction
- Nash Bargaining Solution

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- Shapley Value

Introduction

Basic ingredients of non-cooperative games:

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- Outcome of the game = formed coalitions ($\rightarrow$ partition of the set of players) and actions of coalitions

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Basic ingredients of cooperative games:

- Actions of coalitions (groups of individuals)
- Outcome of the game = formed coalitions ($\rightarrow$ partition of the set of players) and actions of coalitions
- Players' preferences over outcomes (as in non-cooperative games)

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➥ stability (in general), as in non-cooperative games, but towards **groups** of players

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➥ stability (in general), as in non-cooperative games, but towards **groups** of players

Contrary to non-cooperative games, no detail is given on how groups form and make decisions

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  $d = (u_1(D), u_2(D))$: pair of disagreement payoffs

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Example. Exchange economy. Disagreement point $\sim$ initial endowments

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Definition. A bargaining solution is a function ψ that associates with every bargaining problem $(\mathcal{U}, d)$ a unique member $\psi(\mathcal{U}, d)$ of $\mathcal{U}$

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❖ **Pareto optimality (PAR).** For every bargaining problem $(\mathcal{U}, d)$, the bargaining solution $\psi(\mathcal{U}, d)$ is not Pareto dominated by a pair (v_1, v_2) of $\mathcal{U}$: $\nexists\ (v_1, v_2) \in \mathcal{U}$ s.t. $v_i \geq \psi_i(\mathcal{U}, d)$, $i = 1,\ 2$, with at least one strict inequality

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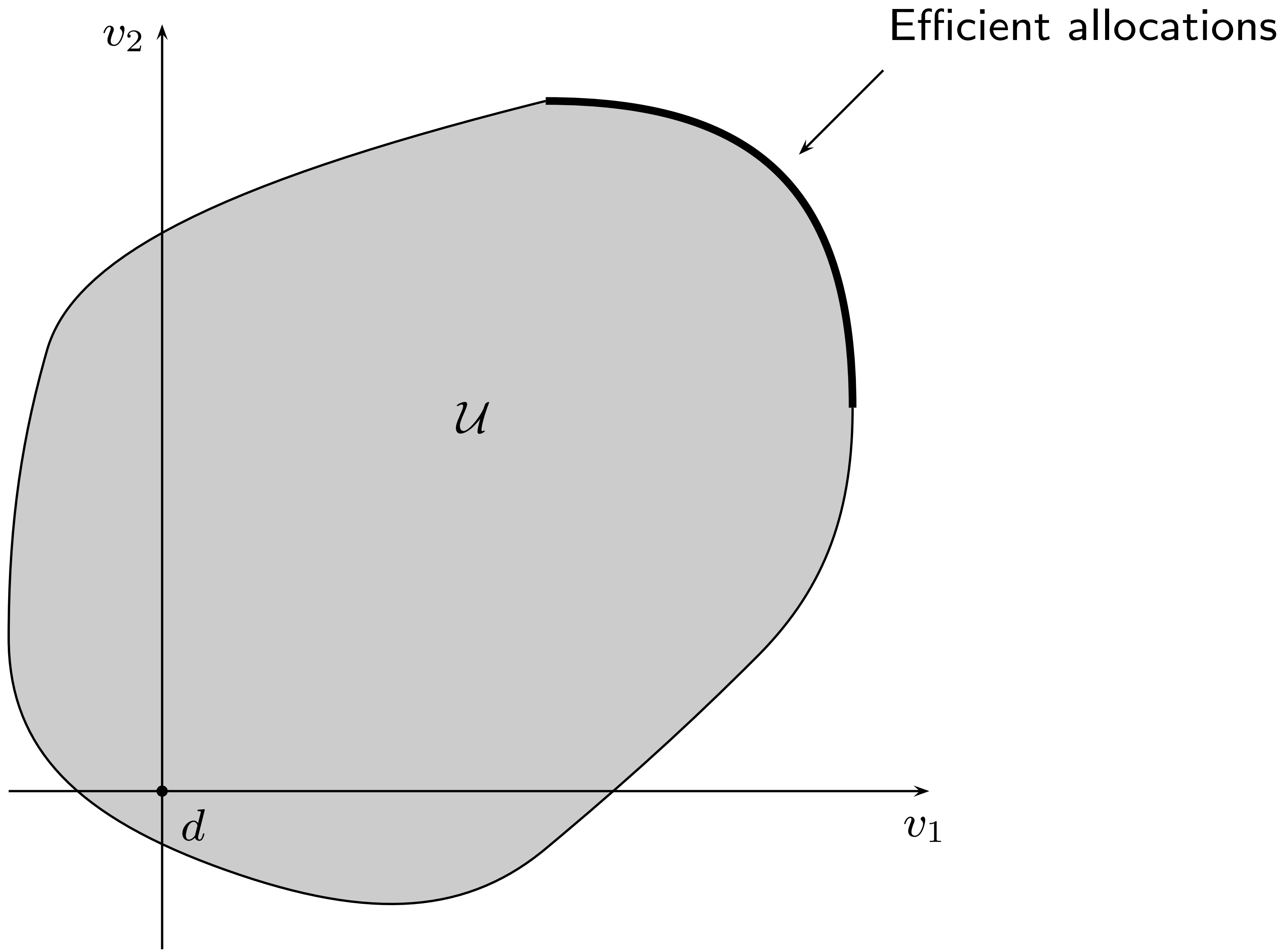
➥ No possible renegotiation improving both players' payoffs


Efficient allocations
v_2
$\mathcal{U}$
d
v_1

❖ **Symmetry (SYM).** ("Equity") If the bargaining problem $(\mathcal{U}, d)$ is symmetric, i.e., $(v_1, v_2) \in \mathcal{U} \Leftrightarrow (v_2, v_1) \in \mathcal{U}$ (the 45° line is a line of symmetry of $\mathcal{U}$) and $d_1 = d_2$, then the bargaining solution gives every player the same payoff: $\psi_1(\mathcal{U}, d) = \psi_2(\mathcal{U}, d)$

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➥ These two axioms give a unique solution for symmetric games

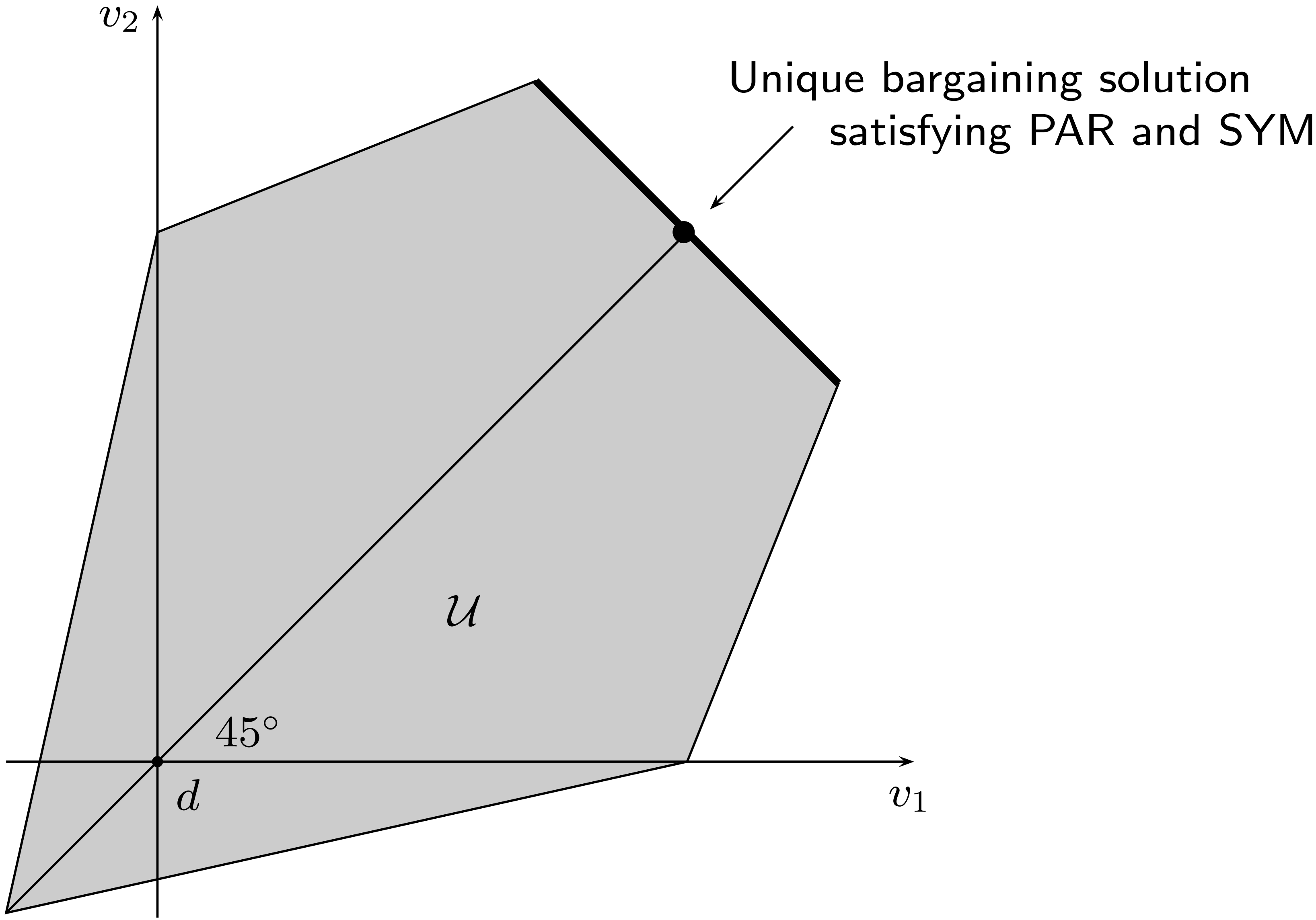
v_2
Unique bargaining solution
satisfying PAR and SYM
$\mathcal{U}$
45°
d
v_1

Figure 1:

✥ **Invariance to equivalent payoff representations (INV).** If the bargaining problem $(\mathcal{U}', d')$ is derived from another bargaining problem $(\mathcal{U}, d)$ by an increasing affine transformation ($v_i' = \alpha_i\, v_i + \beta_i$ and $d_i' = \alpha_i\, d_i + \beta_i$, $i = 1,\, 2$, $\alpha_i > 0$), then the solution of the transformed problem for player i is the transformation of the solution of the original problem:

$$\psi_i(\mathcal{U}', d') = \alpha_i\, \psi_i(\mathcal{U}, d) + \beta_i \quad (i = 1,\, 2)$$

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- ➥ Consistency with the cardinality of expected utility functions
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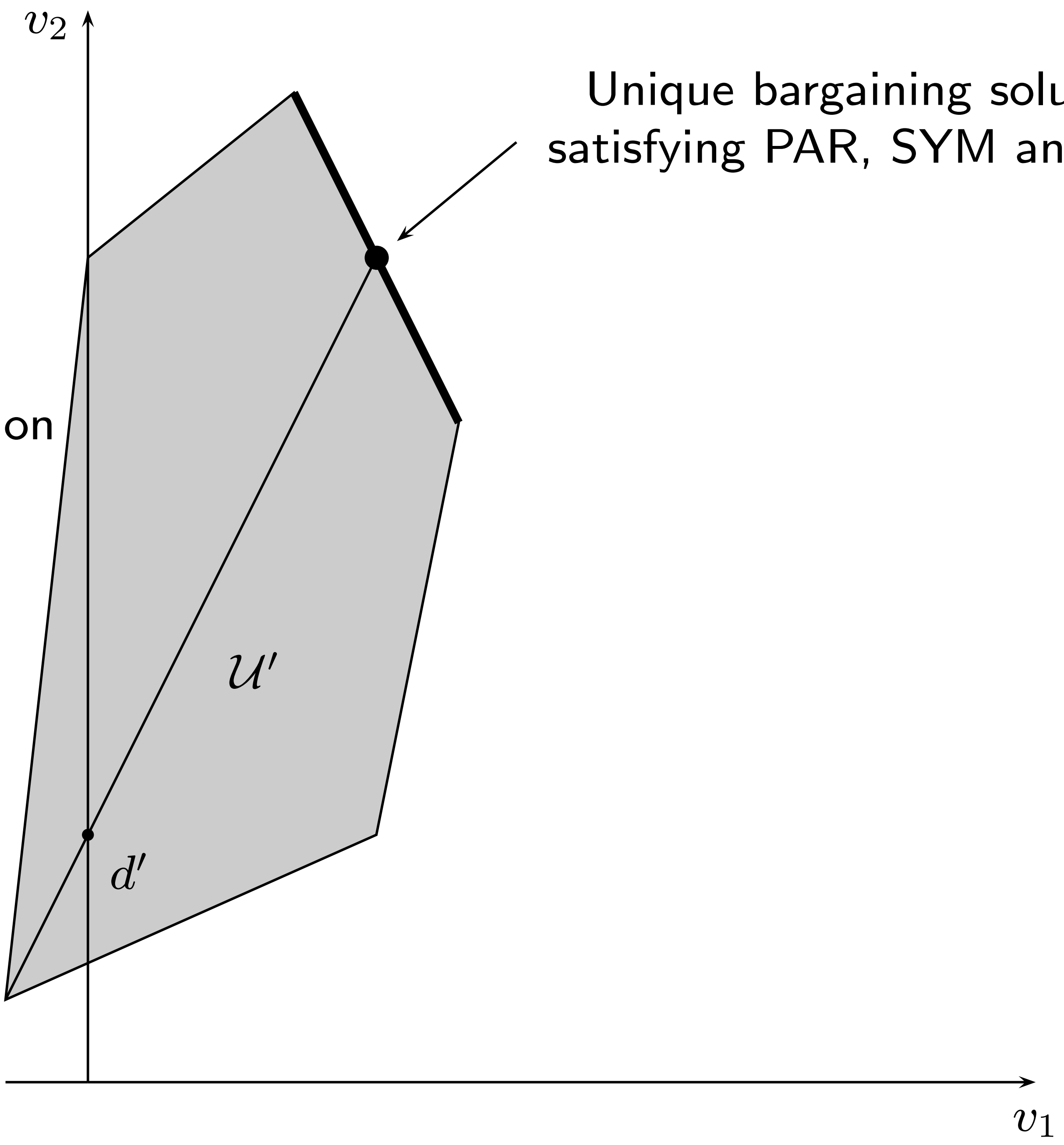
$\Rightarrow$ With these three axioms we get a unique solution for every bargaining problem that can be obtained as a linear transformation of a symmetric bargaining problem



Monotone affine transformation
of the problem of figure 1

$$v_1' = \frac{1}{2}\, v_1$$

$$v_2' = v_2 + 30$$

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✤ **Independence of irrelevant alternatives (IIA).** (invariance to contraction) If two bargaining problems $(\mathcal{U}, d)$ and $(\mathcal{U}', d)$ with the same disagreement point are such that $\mathcal{U} \subseteq \mathcal{U}'$ and $\psi(\mathcal{U}', d) \in \mathcal{U}$ then $\psi(\mathcal{U}, d) = \psi(\mathcal{U}', d)$

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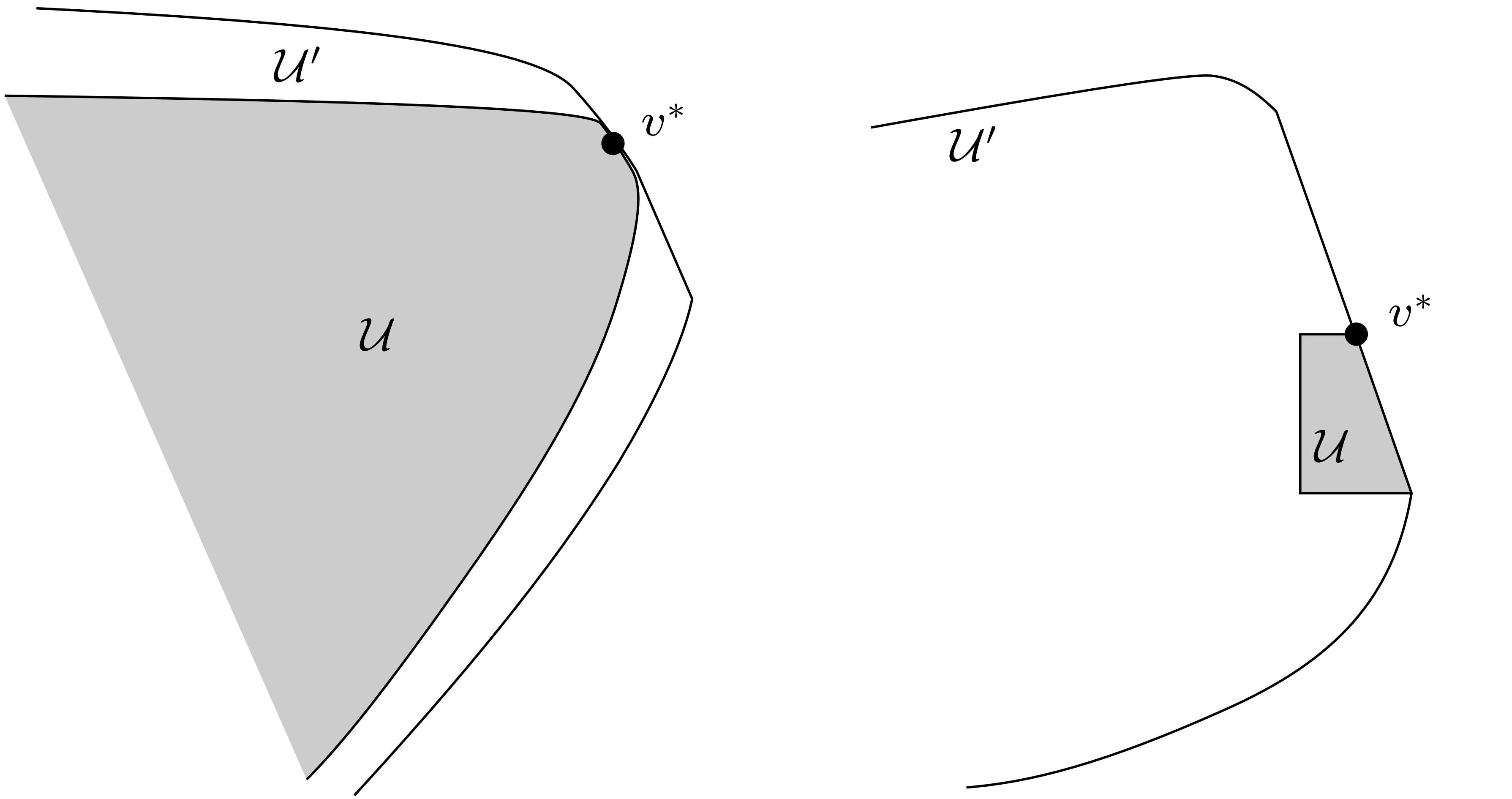
Remark. If ψ is obtained by maximizing a function on $\mathcal{U}$ then this axiom is satisfied



If $\mathcal{U} \subseteq \mathcal{U}'$ and $\psi(\mathcal{U}', d) = v^* \in \mathcal{U}$ then $\psi(\mathcal{U}, d) = v^*$

Proposition. (Nash Theorem) *One and only one bargaining solution satisfies the four axioms PAR, SYM, INV and IIA. It is the* **Nash bargaining solution***, that assigns to every bargaining problem $(\mathcal{U}, d)$ the pair of payoffs that maximizes the Nash product:*

$$\max_{v} \ (v_1 - d_1)(v_2 - d_2) \quad \textit{s.t.} \quad v \in \mathcal{U} \ \ \textit{and} \ \ v \geq d$$

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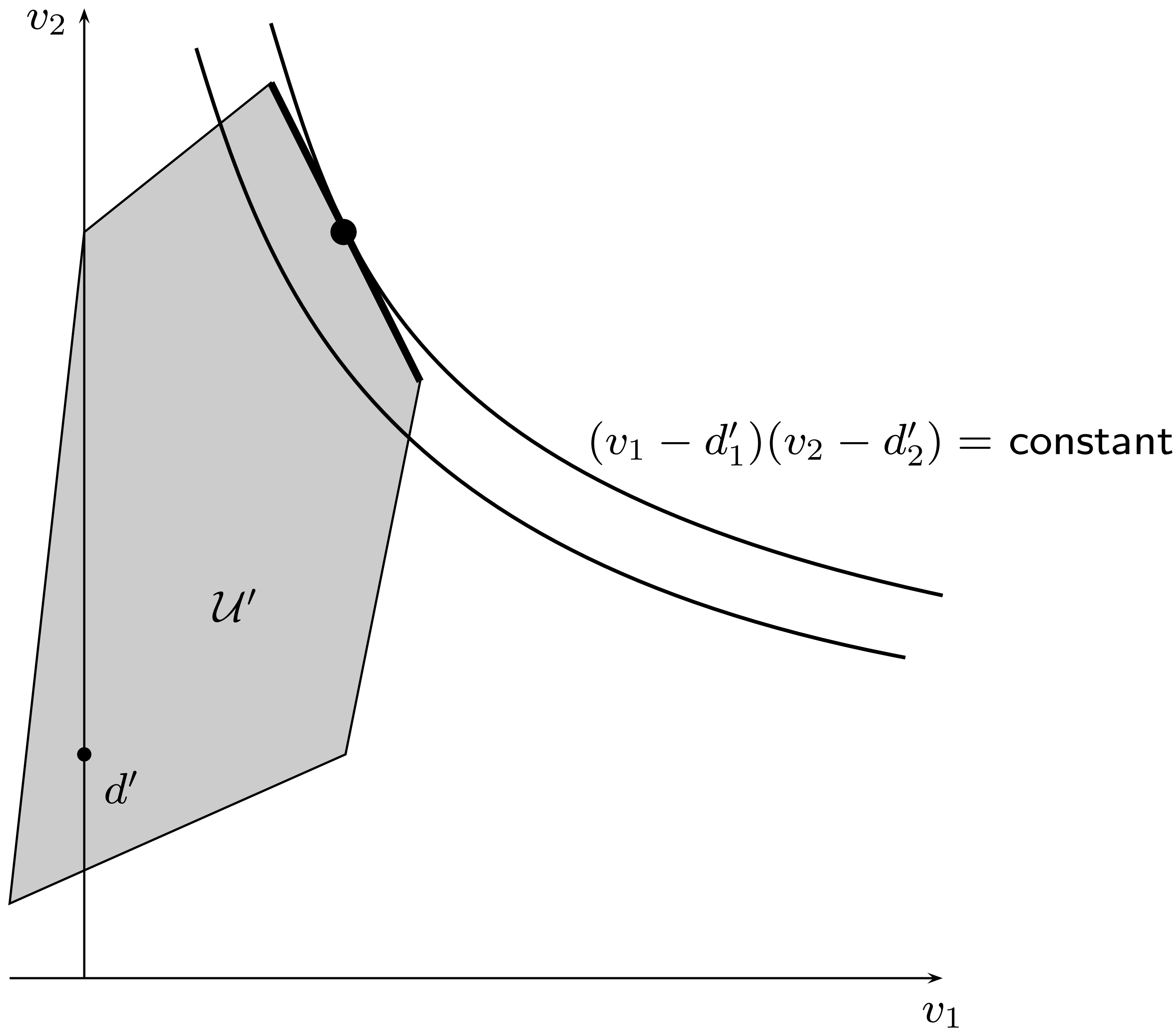
For any value of c, the set of points (v_1, v_2) such that

$$(v_1 - d_1)(v_2 - d_2) = c$$

is an hyperbola $\Rightarrow$ the Nash solution is the pair (v_1, v_2) in $\mathcal{U}$ on the highest such hyperbola


v_2
v_1
$(v_1 - d'_1)(v_2 - d'_2) =$ constant
$\mathcal{U}'$
d'

Intuition for the proof of uniqueness.

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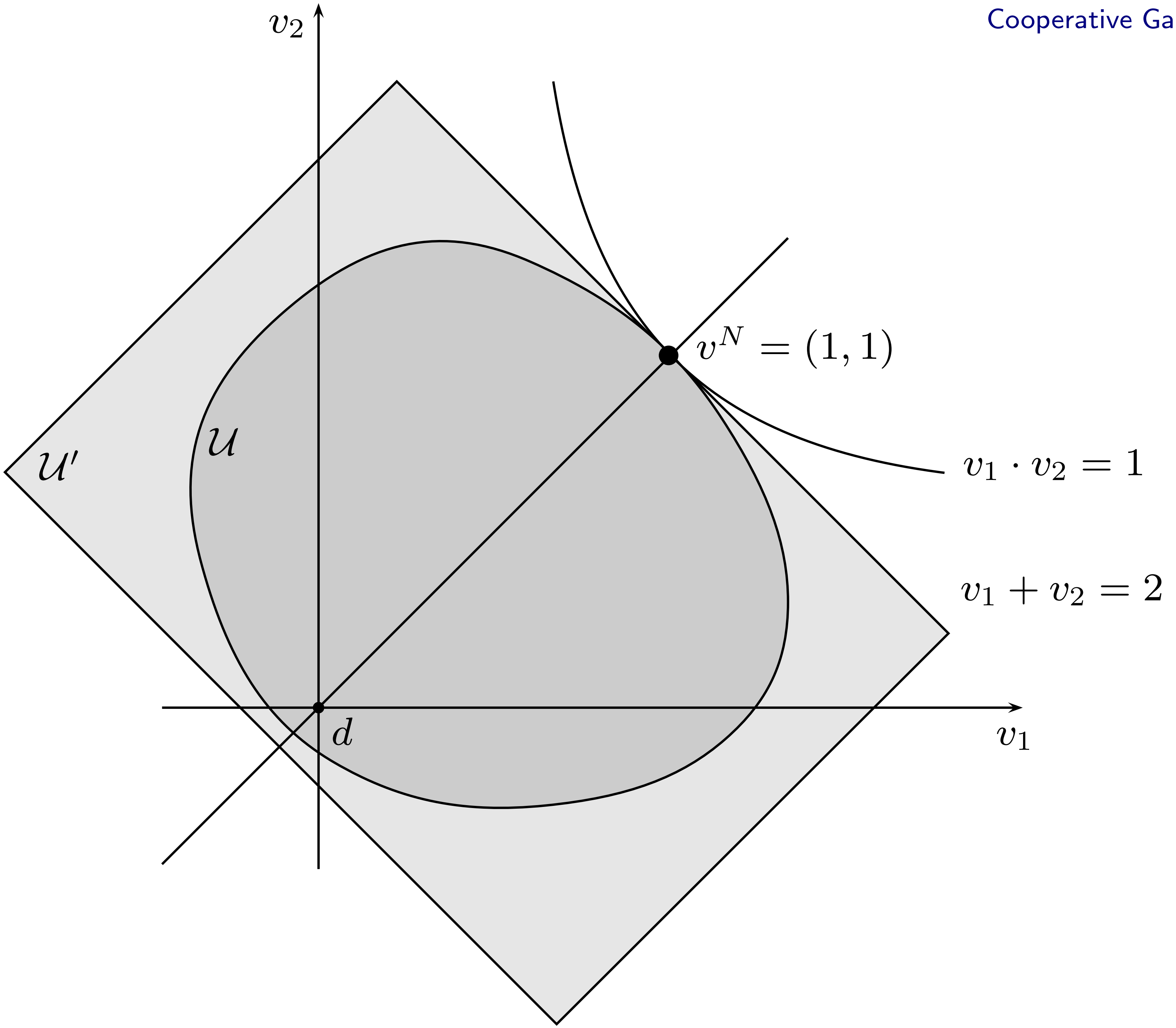
v_2
$v^N = (1,1)$
$\mathcal{U}$
$\mathcal{U}'$
$v_1 \cdot v_2 = 1$
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d
v_1

Link with the Strategic Approach

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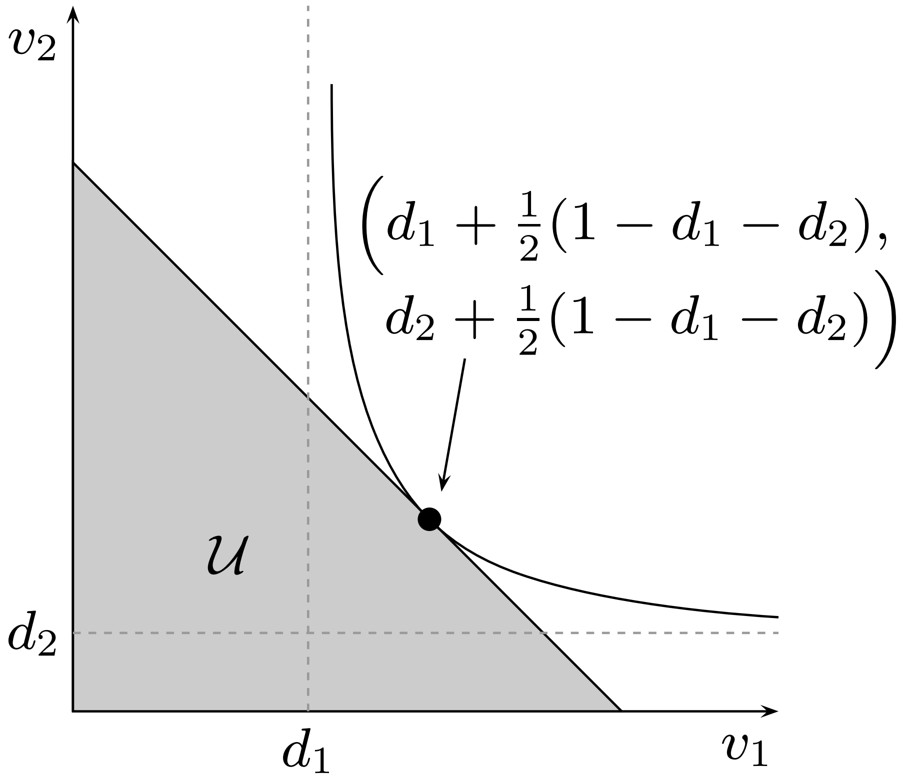

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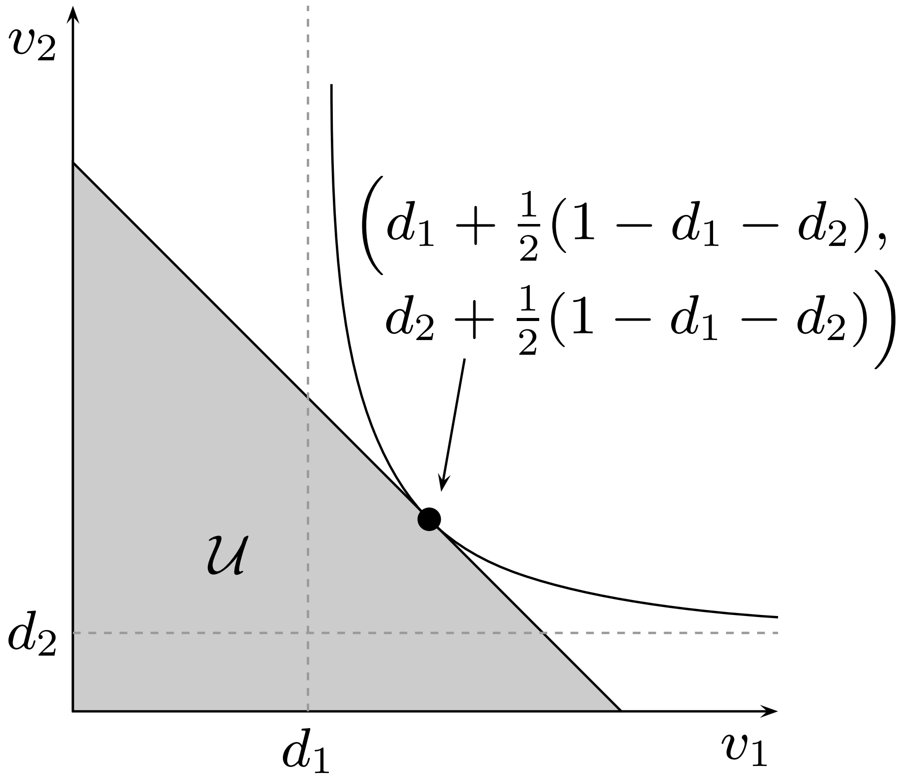

☞ The Nash solution is the SPNE outcome of the alternating offers bargaining game with risk of breakdown $\alpha \to 0$ (without discounting), where $d = b$ is the pair of payoffs when negotiations terminate (Binmore et al., 1986)

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- **2nd solution**: taking into coalitions formations, or at least the potential threat of coalitions formations

Coalitions and Characteristic Functions

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$S = \{i\}$: coalition of one player (singleton)

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A coalition is a subset of players $S \subseteq N \equiv \{1, \ldots, n\}$, $S \neq \emptyset$ ($2^n - 1$ possible coalitions)

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✍ Find a superadditive game which is not monotonic

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A player j is a dictator if a coalition is winning iff player j belongs to it ($v(S) = 1 \Leftrightarrow j \in S$)

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Definition. A payoff profile $(x_i)_{i \in N}$ is S-feasible if $x(S) = v(S)$. It is feasible if it is N-feasible

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Remark. Collective rationality ($x(N) = v(N)$) and individual rationality ($x_i \geq v(i)\ \forall\, i$) are satisfied

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➥ No difference between veto and dictatorship. (The Shapley value will make a difference)

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(ii) Let $V \neq \emptyset$ be the set of veto players and x a positive and feasible allocation giving zero payoff to all non-veto players:

$$\left.\begin{aligned} x_i &\geq 0 \quad \forall\, i \in V \\ x_i &= 0 \quad \forall\, i \notin V \\ \sum_{i \in N} x_i &= 1 \end{aligned}\right\} \tag{1}$$

- If S is winning then $V \subseteq S$, so $x(S) = 1 = v(S)$
- If S is loosing then $v(S) = 0$, so $x(S) \geq v(S)$

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$j \notin V \overset{\text{def}}{\Rightarrow} \exists\, S$, $j \notin S$, s.t. $v(S) = 1 > x(S)$, so S blocks x, i.e. $x \notin$ core $\square$

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General necessary and sufficient conditions for the core to be non-empty: Bondareva (1963) and Shapley (1967) (see Osborne and Rubinstein, 1994, pp. 262–263)

A Production Economy

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Firm (landowner): player 0

K workers: players $1, \ldots, K$

k workers with the landowner can produce $f(k) \geq 0$, where $f \nearrow$, concave and $f(0) = 0$. Without the landowner they produce nothing

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$$\Rightarrow \quad \begin{cases} N = \{0, 1, \ldots, K\} \\ v(S) = \begin{cases} 0 & \text{if } 0 \notin S \\ f(|S| - 1) & \text{if } 0 \in S \end{cases} \end{cases}$$

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$$x_0 + x_1 + \cdots + x_K = f(K) \tag{2}$$

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(4) $\Rightarrow x(N \backslash i) \geq f(K-1)\ \forall\, i \neq 0 \overset{(2)}{\Rightarrow} f(K) - x_i \geq f(K-1) \Rightarrow$
$x_i \leq f(K) - f(K-1)\ \forall\, i \neq 0$

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If $0 \in S$ then $x_i \leq f(K) - f(K-1)\ \forall\, i \in N \backslash S \Rightarrow$
$x(N \backslash S) \leq (K-k)(f(K) - f(K-1))$, where $k = |S| - 1 =$ nb of workers in S

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$$x_0 + x_1 + \cdots + x_K = f(K) \tag{2}$$

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Let us show the converse: let x be in this set

If $0 \notin S$ then $v(S) = 0$ so $x(S) \geq v(S)$

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Conclusion: Each worker obtains at best his marginal productivity when all workers are employed, and the landowner gets the remaining payoff

Unionized Workers

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$\Rightarrow$ core $= \{(x_0, x_1, \ldots, x_K) : x_i \geq 0 \; \forall \; i, \; \sum x_i = f(K)\}$

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1 player, $i = 3$, has a right shoe

$v(S) = 1$ € for each pairs of shoes that coalition S can obtain

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The Shapley value gives slightly more than 0.5 for right shoes and slightly less than 0.5 for left shoes

Shapley Value

(September 3, 2007)

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Classical solution concept for n-person cooperative games with transferable utility (TU games)

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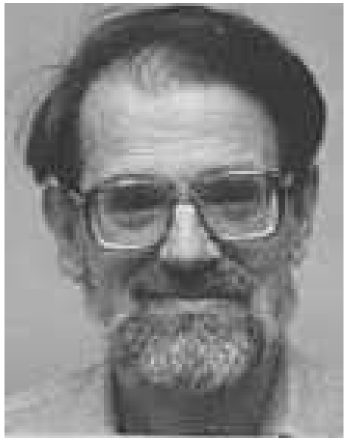

Figure 2: Lloyd Shapley (1923–)

Like the Nash bargaining solution, the Shapley (1953) value is a solution concept satisfying some reasonable axioms ($+$ existence and uniqueness)

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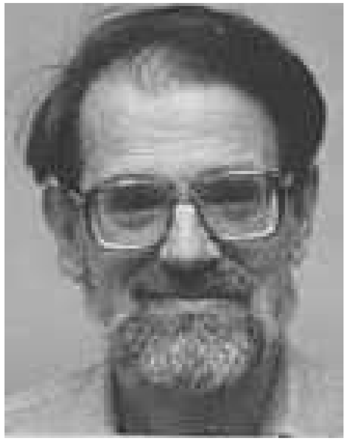

Figure 2: Lloyd Shapley (1923–)

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Appropriate solution concept for problems of cost sharing or allocation of resources (telecommunications, joint ownership, . . .)

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$\varphi_i(v)$ is a power index for player i / a value of the game for player i

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❖ **Axiom 2. Symmetry (SYM).** If i and j are symmetric (substitutes), i.e.,

$$v(S \cup \{i\}) = v(S \cup \{j\}) \quad \forall\, S \not\ni i, j$$

$$\text{then} \quad \varphi_i(v) = \varphi_j(v)$$

❖ **Axiom 3. Null player (NUL).** If i is null, i.e.,

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❖ **Axiom 4. Linearity (LIN).** Define $(v + w)(S) = v(S) + w(S)$. Then,

$$\varphi(v + w) = \varphi(v) + \varphi(w)$$

(mathematical simplification, but no clear interpretation)

Shapley Theorem. There exists one and only one solution φ satisfying the four preceding axioms. It can be calculated explicitly:

$$\varphi_i(v) = \frac{1}{n\,!} \sum_R \left[v(S_i^R \cup \{i\}) - v(S_i^R) \right]$$

where the sum (R) is over all $n\,!$ permutations of N

and $S_i^R \subseteq N$ is the coalition of players preceding i in order R $(v(\emptyset) = 0)$

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PAR + NUL $\Rightarrow \varphi_1(v) = \varphi_3(v) = 0$ and $\varphi_2(v) = 1$

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$3\,! = 6$ possible orders	Marginal contributions of player 2
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132	$v(132) - v(13) = 1$
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Electoral games and political power

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Quota $Q = 1/2$ (simple majority)

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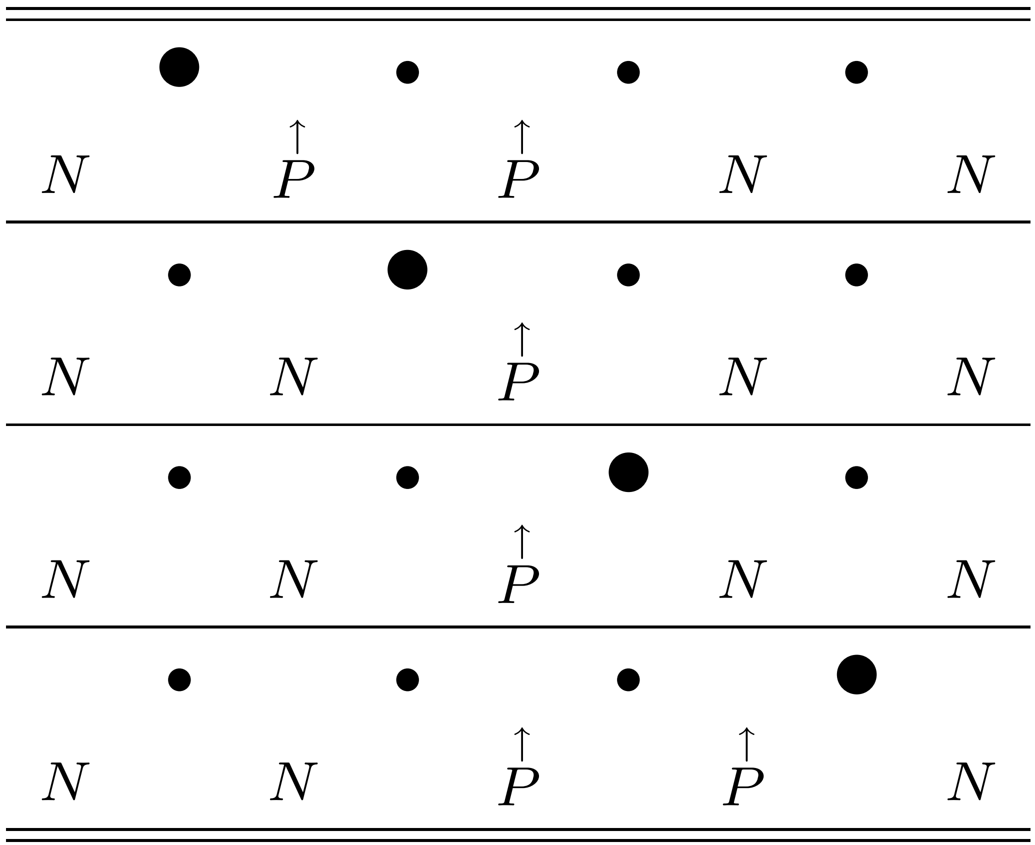
4 equally likely order configurations for a large party, with 5 equally likely positions in each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$$
\begin{array}{ccccccccc}
\hline\hline
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N & & \overset{\uparrow}{P} & & \overset{\uparrow}{P} & & N & & N \\
\hline
 & \bullet & & \Large\bullet & & \bullet & & \bullet & \\
N & & N & & \overset{\uparrow}{P} & & N & & N \\
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N & & N & & \overset{\uparrow}{P} & & N & & N \\
\hline
 & \bullet & & \bullet & & \bullet & & \Large\bullet & \\
N & & N & & \overset{\uparrow}{P} & & \overset{\uparrow}{P} & & N \\
\hline\hline
\end{array}
$$

$$\Rightarrow \quad \varphi_1(v) = \varphi_2(v) = 6/20 = 3/10 < q_1 = q_2 = 1/3$$

	⬤		•		•		•	
N		$\overset{\uparrow}{P}$		$\overset{\uparrow}{P}$		N		N
	•		⬤		•		•	
N		N		$\overset{\uparrow}{P}$		N		N
	•		•		⬤		•	
N		N		$\overset{\uparrow}{P}$		N		N
	•		•		•		⬤	
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$$\Rightarrow \quad \varphi_1(v) = \varphi_2(v) = 6/20 = 3/10 < q_1 = q_2 = 1/3$$

$$\Rightarrow \quad \varphi(v) = \left(\frac{3}{10}, \frac{3}{10}, \frac{4}{30}, \frac{4}{30}, \frac{4}{30}\right)$$

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No, because the game would be symmetric $\Rightarrow$ small parties would share $1/3$ instead of $1/2$

Paradox of the new members of the European union council

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Members	Weight	Shapley Val.	Weight	Shapley Val.
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Belgium	2	0.150	5	0.081
Nethederlands	2	0.150	5	0.081
Luxembourg	1	0.000	2	0.010
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Luxembourg: null player in 1958. In 1973, relative weight $\searrow$ but power $\nearrow$

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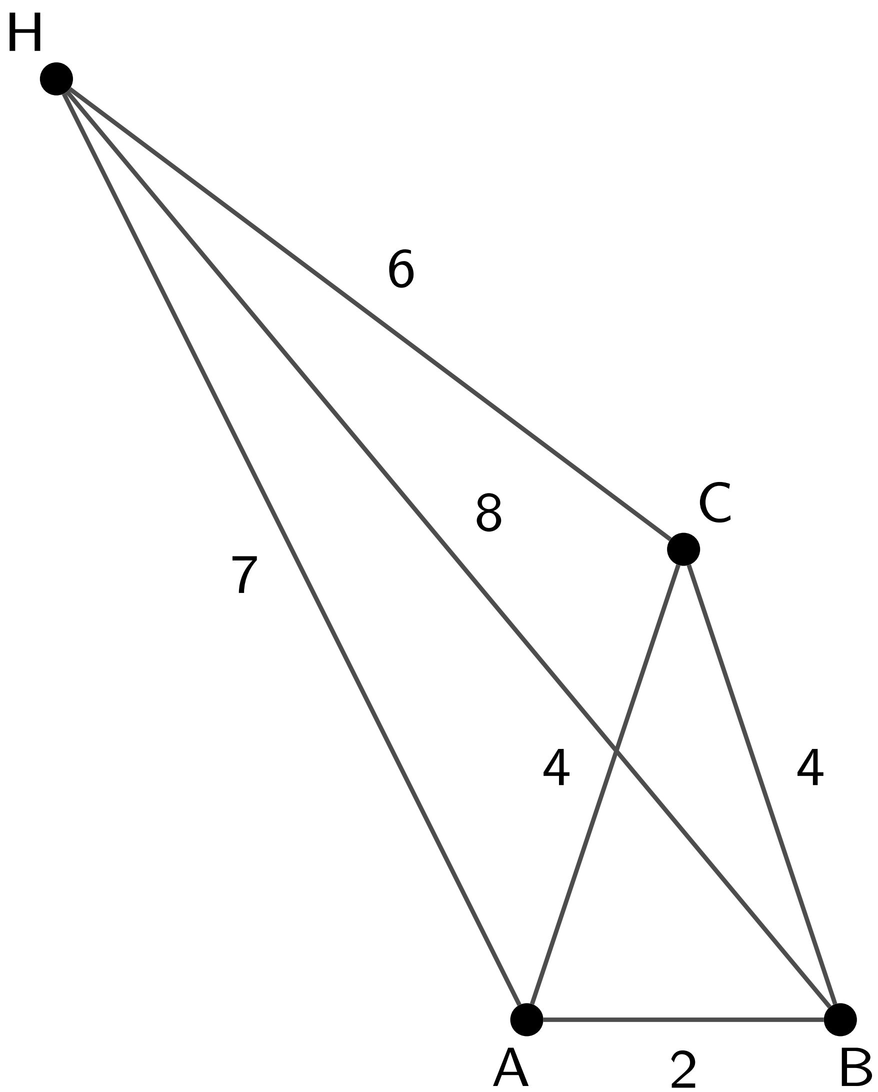

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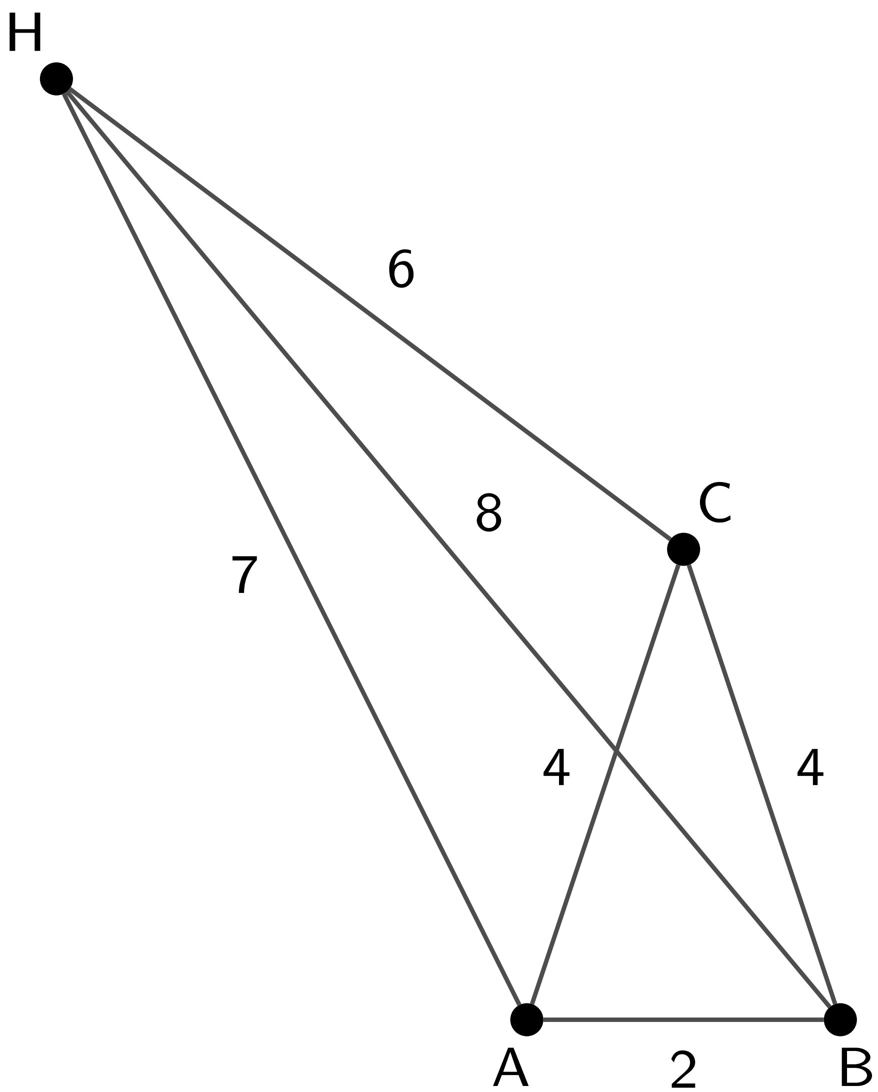

$$v(A) = 20 - 14 = 6$$
$$v(B) = 4$$
$$v(C) = 8$$

$$v(AB) = 23$$
$$v(AC) = 23$$
$$v(BC) = 22$$
$$v(ABC) = 60 - 19 = 41$$

Possible orders	Marginal Contributions A	B	C
ABC	6	17	18
ACB	6	18	17
BAC	19	4	18
BCA	19	4	18
CAB	15	18	8
CBA	19	14	8
$\sum_R$	84	75	87
$\varphi = \sum_R / n!$	14	12.5	14.5
Cost allocation	6	7.5	5.5

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$$\Rightarrow \quad \beta = \left(\frac{1}{5}, \frac{3}{5}, \frac{1}{5}\right) \neq \varphi = \left(\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right)$$

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