Dynamic Games of Incomplete Information

Equilibrium Refinement and Signaling Games

Outline

(November 20, 2007)

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- Introductory Examples
- Sequential Rationality and Perfect Bayesian Equilibrium
- Strong Belief Consistency and Sequential Equilibrium
- Signaling Games
- Application: Spence's (1973) Model of Education

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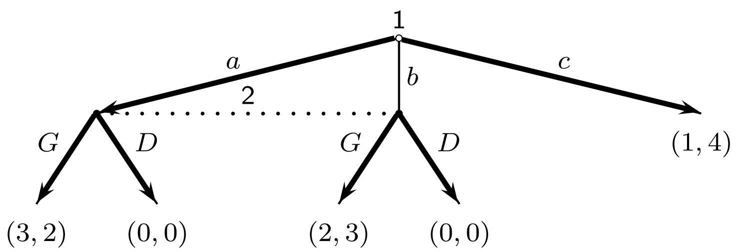
In games with imperfect information, subgame perfection is not always strong enough to eliminate "irrational decisions" or "incredible threats" off the equilibrium path

Example.



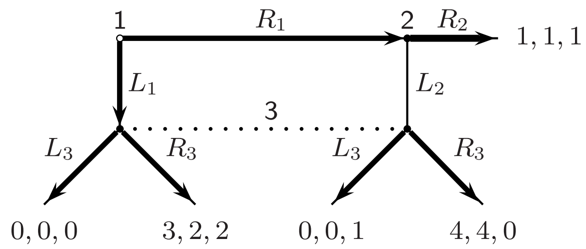
(c, D) is a (SP)NE but D is not an optimal decision at player 2's information set

Sequential rationality $\sim$ generalization of backward induction

➥ Require rational decisions even at information sets off the equilibrium path (even if they are not singleton information sets)

$\Rightarrow$ Player 2 plays G $\Rightarrow$ Player 1 plays a

Example. (Selten's (1975) "horse")



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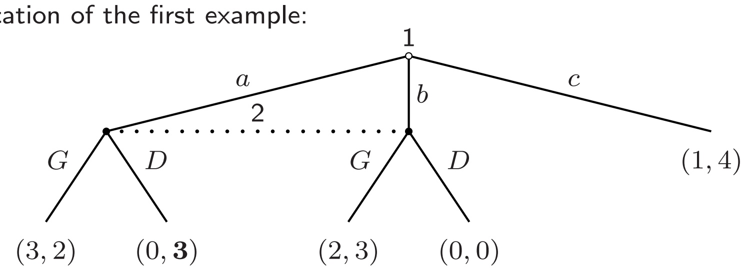
2 pure strategy (SP)NE: (R_1, R_2, L_3) and (L_1, R_2, R_3)

But in (L_1, R_2, R_3) the action R_2 of player 2 is not sequentially rational given that player 3 plays R_3 $(4 > 1)$

In the previous examples we have eliminated SPNE in which the action of some player is never optimal, **whatever his belief about past play**

Modification of the first example:



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➠ If player 1 plays c, sequential rationality of player 2 is not well defined (playing G or playing D?)

➠ The strategy profile is usually not sufficient to define sequential rationality

➠ The *solution concept* is not only characterized by a *strategy profile* but also by a **belief system**

Belief System

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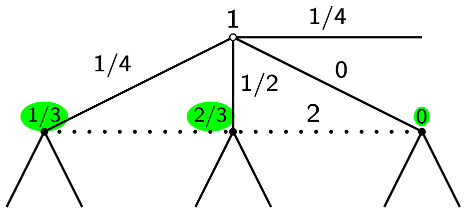
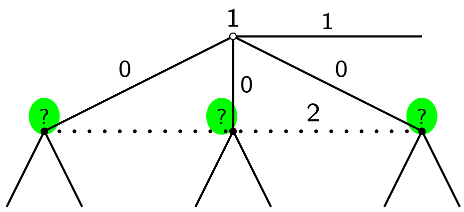


➥ Bayes' rule can be applied: $\mu_2 = (\frac{1}{3}, \frac{2}{3}, 0)$



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➥ Bayes' rule cannot be applied: $\mu_2 = ?$ (divide by zero)

Belief system: collection of probability distributions on decision nodes, one distribution for each information set

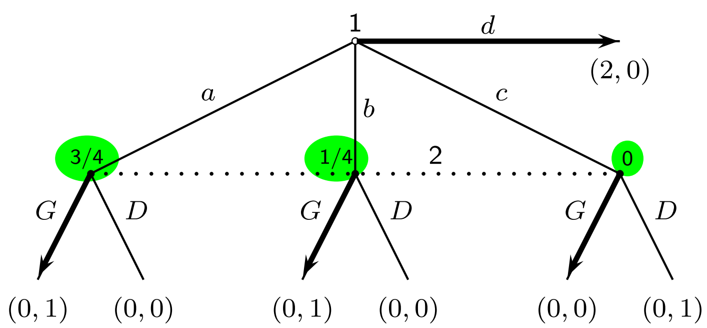
☞ trivial in perfect information games (probability 1 at every node)

A pair (σ, μ), where σ is a profile of behavioral strategies and μ a belief system, is a **weak sequential equilibrium**, or **perfect Bayesian equilibrium** (PBE), if

- **Sequential Rationality.** For every player i and every information set of player i, the local strategy of player i at this information set maximizes his expected utility given his belief at this information set and the strategies of the other players

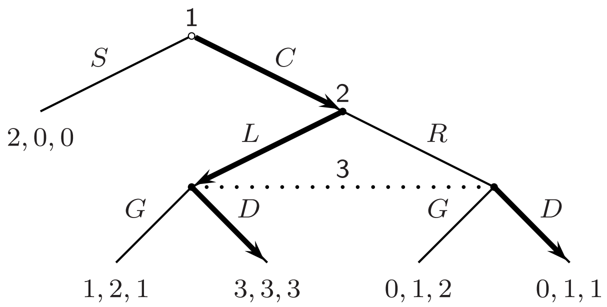
- **Weak Belief Consistency.** In every subgame (along and off the equilibrium path), beliefs are computed by Bayes' rule according to σ when it is possible. When Bayes' rule cannot be applied, beliefs can be chosen arbitrarily

Example. (d, G) is a perfect Bayesian equilibrium (PBE)



Remark. Many other belief systems are possible ($(1, 0, 0)$, $(0, 1, 0)$, $(1/3, 1/3, 1/3)$, ...)

Example. (Belief consistency in subgames off the equilibrium path)



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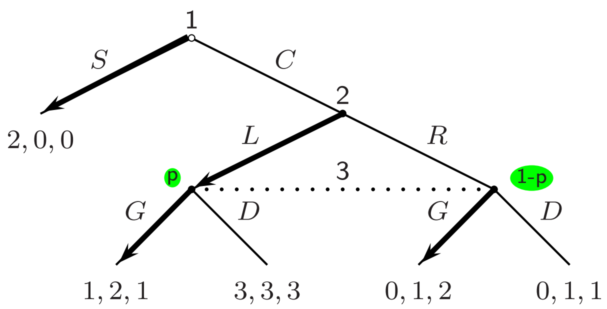
Unique SPNE: (C, L, D)

Bayes' rule can be applied everywhere

Sequential rationality is satisfied

Next, consider the Nash equilibrium (S, L, G) (which is not subgame perfect)



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Bayes' rule does not apply for player 3 *in the entire game*

Consider the belief $\mu_3 = (p, 1-p)$ of player 3, with $p < 1/3$

$\Rightarrow$ Sequential rationality is satisfied ($G \xrightarrow{3} 2 - p > 5/3, \quad D \xrightarrow{3} 1 + 2p < 5/3$)

But μ_3 is not weakly consistent because *in the strict subgame* (off the equilibrium path) Bayes' rule implies $p = 1$

Strong Belief Consistency

Strictly positive strategy of player i : $\sigma_{h_i}(a_i) > 0$ for every action available at information set h_i of player i, $a_i \in A(h_i)$, and for every information set of player i, $h_i \in H_i$

Strong belief consistency: there is a sequence $\{(\tilde{\sigma}^k, \tilde{\mu}^k)\}_k$, such that

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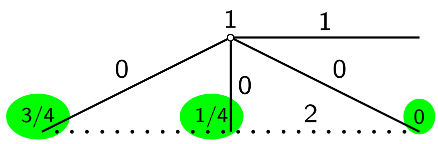
$$\lim_{k \to \infty} (\tilde{\sigma}^k, \tilde{\mu}^k) = (\sigma^*, \mu^*)$$

where

- $\tilde{\sigma}^k$ is a strictly positive strategy profile
- $\tilde{\mu}^k$ is obtained by Bayes' rule from $\tilde{\sigma}^k$

Strong sequential equilibrium (SE): Sequential rationality + **strong** belief consistency

Example.



$$\tilde{\sigma}_1^k = (3\varepsilon^k, \varepsilon^k, (\varepsilon^k)^2, 1 - 4\varepsilon^k - (\varepsilon^k)^2) \longrightarrow \sigma^* = (0, 0, 0, 1)$$

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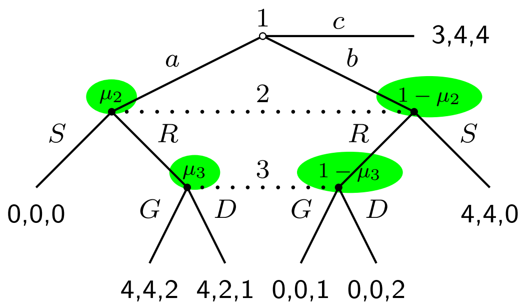
$$\tilde{\mu}_2^k = \left(\frac{3\varepsilon^k}{4\varepsilon^k + (\varepsilon^k)^2}, \frac{\varepsilon^k}{4\varepsilon^k + (\varepsilon^k)^2}, \frac{(\varepsilon^k)^2}{4\varepsilon^k + (\varepsilon^k)^2} \right) \longrightarrow \mu^* = \left(\frac{3}{4}, \frac{1}{4}, 0 \right)$$

Remark Strong belief consistency requires finite action sets and state spaces (except in the last decision nodes of the game tree)

Example of a PBE which is not a (strong) sequential equilibrium



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Consider the (SP)NE $(c, (\frac{1}{2}S + \frac{1}{2}R), (\frac{1}{2}G + \frac{1}{2}D))$

Sequential rationality: Player 1 a and $b \longrightarrow 2$, $c \longrightarrow 3 \geq 2$ OK

Player 2 $\begin{cases} S \to 4 - 4\mu_2 \\ R \to 3\mu_2 \end{cases} \Rightarrow \mu_2 = 4/7$ Player 3 $\begin{cases} G \to 1 + \mu_3 \\ D \to 2 - \mu_3 \end{cases} \Rightarrow \mu_3 = 1/2$

But $\mu_2 = 4/7 \neq \mu_3 = 1/2$ is not strongly consistent: for every perturbed strategy profile $\tilde{\sigma}^k$ we have $\lim_\infty \tilde{\mu}_2^k = \lim_\infty \tilde{\mu}_3^k$

Proposition 1 *Every finite extensive form game has at least one (possibly mixed) sequential equilibrium, and therefore at least one PBE*

Proposition 2 *The set of sequential equilibria is included in the set of SPNE*

More generally, we have:

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$$\{SE\} \subseteq \{PBE\} \subseteq \{SPNE\} \subseteq \{NE\}$$

Proposition 3 *In games with perfect information the set of sequential equilibria (weak and strong) coincides with the set of SPNE*

Remark There exist stronger versions of perfect Bayesian equilibrium than those presented here, which apply to more specific dynamic games. For example, in some classes of multistage games with independent types, Fudenberg and Tirole (1991) define a perfect Bayesian equilibrium (without referring to perturbed strategies) which is equivalent to the (strong) sequential equilibrium

A particularly simple class of dynamic games of incomplete information in which the
15/ simplest version of PBE and the strong SE coincide is the class of signaling games

Signaling Games

- Two players: the **sender** (player 1) and the **receiver** (player 2).
- States of the world: **types** T of player 1
- Prior probability distribution over types: $p \in \Delta(T)$
- Player 1 observes his type $t \in T$ and chooses an action (**message**, **signal**) $m \in M$

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- Afterward, player 2 observes the message m (but not the type t of player 1) and chooses an action (**response**) $r \in R$
- The game ends with payoffs $u_1(m, r; t)$ and $u_2(m, r; t)$

➥ Strategies: $\sigma_1 : T \to \Delta(M)$ and $\sigma_2 : M \to \Delta(R)$

Remark The set of messages available to the sender may depend on his type, $M(t)$, and the set of responses of the receiver may depend on the message, $R(m)$

Remark If $u_1(m, r; t)$ and $u_2(m, r; t)$ do not depend on m the game is also called a **costless communication game**, or **cheap talk game**

17/ If, in addition, the set of messages M depends on the type t of the sender, the game is called a communication game with **certifiable** or **verifiable information**, or **persuasion game**

Ex: If $M(t_1) = \{m_1, \overline{m}\}$ and $M(t_2) = \{m_2, \overline{m}\}$, then m_i = certificate/proof that the sender's type is t_i

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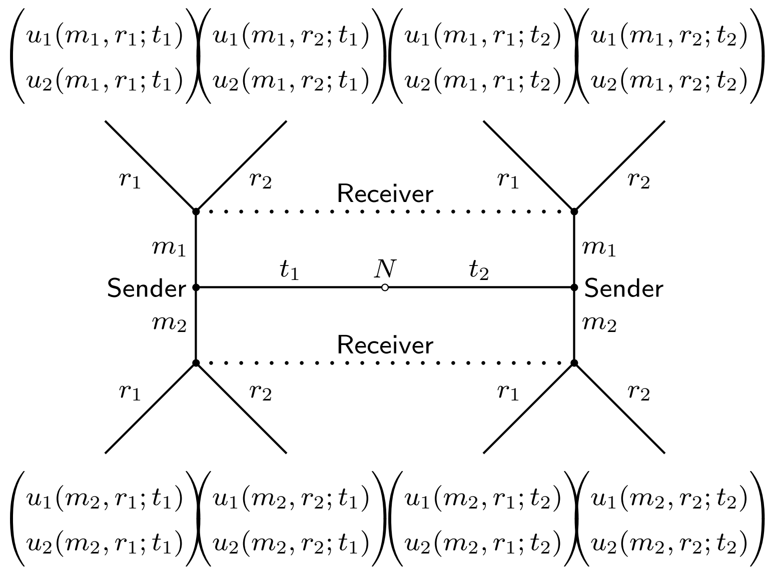
Binary case: 2 types / 2 messages / 2 responses



Perfect Bayesian equilibrium (σ, μ) **of the signaling game:**

(i) **Sequential rationality of player 1.** $\forall\, t \in T$, $\forall\, m^* \in \text{supp}[\sigma_1(t)]$,

$$m^* \in \arg\max_{m \in M} \sum_{r \in R} \sigma_2(r \mid m)\, u_1(m, r; t)$$

(ii) **Sequential rationality of player 2.** $\forall\, m \in M$, $\forall\, r^* \in \text{supp}[\sigma_2(m)]$,

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$$r^* \in \arg\max_{r \in R} \sum_{t \in T} \mu(t \mid m)\, u_2(m, r; t)$$

(iii) **Belief consistency.** μ is obtained by Bayes rule when possible:

$$\forall\, m \in \text{supp}[\sigma_1], \quad \mu(t \mid m) = \frac{\sigma_1(m \mid t) p(t)}{\sum_{s \in T} \sigma_1(m \mid s) p(s)}$$

✍ Difference with the definition of a Nash equilibrium of the signaling game?

Proposition 4 *In (finite) signaling games the set of perfect Bayesian equilibria coincides with the set of sequential equilibria*

➥ Every belief off the equilibrium path can be obtained as the limit of perturbed beliefs (✍ show this property with 2 types and 2 messages)

Definition An equilibrium is **separating** (**fully revealing** (FR)) if the receiver knows the sender's type when he chooses his response ⇒ degenerated beliefs (1 or 0) after every message sent along the equilibrium path (i.e., in $\text{supp}[\sigma_1]$)

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⇒ the sender sends a different message for each of his type

Definition An equilibrium is **pooling** (**non revealing** (NR)) if the receiver's beliefs are the same as the prior beliefs after every message sent along the equilibrium path (i.e., in $\text{supp}[\sigma_1]$)

⇒ the sender's strategy does not depend on his type

Definition An equilibrium is **partially revealing** (PR) if it is neither fully revealing nor non revealing

Example. $(\Pr(t_1) = \Pr(t_2) = 1/2)$

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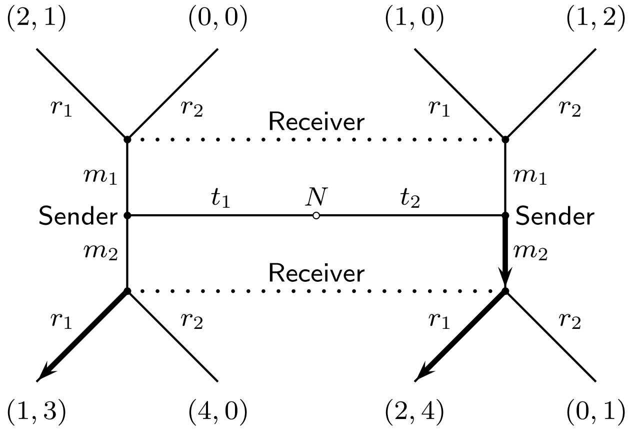

Whatever the receiver's belief after message m_2, his only optimal action is $r_1 \Rightarrow r_1 \mid m_2$ at every PBE $\Rightarrow m_2 \mid t_2$ at every PBE

Existence of a separating equilibrium?

❶ Strategy $m_1 \mid t_1$, $m_2 \mid t_2$ of the sender

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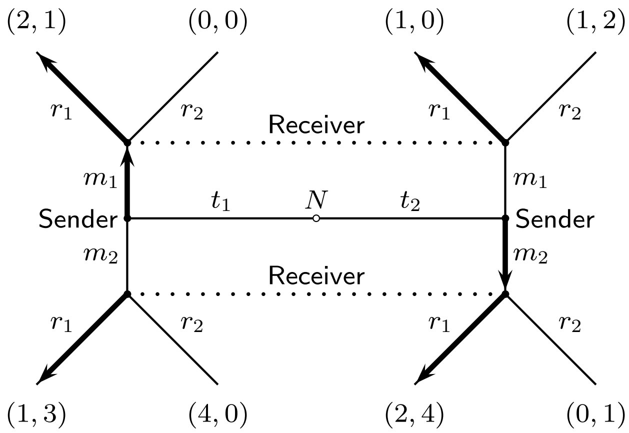

$\Rightarrow$ Belief of the receiver : $\mu_2(t_1 \mid m_1) = \mu_2(t_2 \mid m_2) = 1$

$\Rightarrow$ The receiver plays $r_1 \mid m_1 \Rightarrow$ No profitable unilateral deviation $\Rightarrow$ PBE

Existence of a separating equilibrium?

❷ Strategy $m_2 \mid t_1$, $m_1 \mid t_2$ of the sender

This is not an equilibrium (see above, because $m_2 \mid t_2$ at every PBE)

Existence of a pooling?

❸ Strategy $m_1 \mid t_1$, $m_1 \mid t_2$ of the sender

23/ This is not an equilibrium (see above, because $m_2 \mid t_2$ at every PBE)

Existence of a pooling equilibrium?

❹ Strategy $m_2 \mid t_1$, $m_2 \mid t_2$ of the sender

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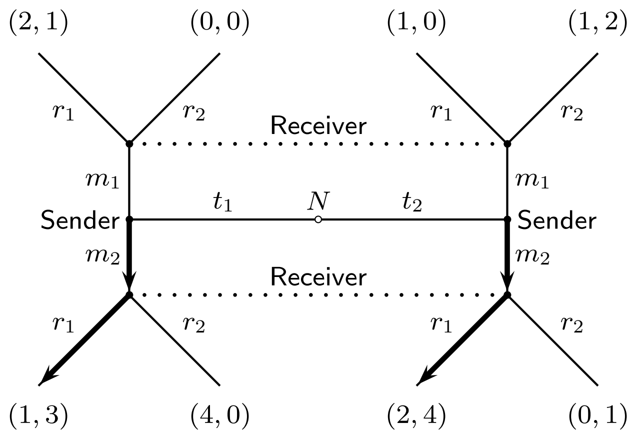

⇒ belief of the receiver : $\mu_2(t_1 \mid m_2) = 1/2$ and $\mu_2(\cdot \mid m_1)$ arbitrary

⇒ t_1 does not deviate if the receiver plays $r_2 \mid m_1$ with probability $\geq 1/2$

⇒ $\mu_2(t_1 \mid m_1)$ should be smaller than 2/3

✍ Write the previous signaling game in normal form and show that the set of pure strategy Nash equilibria coincides with the set of pure strategy PBE

✍ Find a signaling game with a Nash equilibrium outcome which is not included in the set of PBE outcomes

Application: Spence's (1973) Model of Education

Signaling game (message = level of education) from a job candidate to employers (in competition) who don't know the ability (the productivity) of the candidate

Without information, the competitive wage is equal to the average productivity ⇒ high skill workers are underpaid

Spence (1973) has shown how the level of education can be a credible signal of ability/productivity, even when education has no direct impact on workers' productivity

Idea: An agent's desutility (cost) for investing in a higher level of education is smaller for highly productive agents than for less productive agents

⇒ A highly productive agent tends to invest in higher levels of education

⇒ Potential employers understand this, and thus are willing to pay more workers with high levels of education, even if education has no direct impact on productivity

A simple version of the model with two types of workers.

- Sender: **job candidate**
- Receiver: **employers** (perfect competition)
- Types : $T = \{t^H, t^L\}$, $t^H > t^L > 0$ (high / low **ability**) $\Pr(t^H) = p$
- Costly signal (message) of the candidate: level $e \geq 0$ of **education**
- Response of the employers: **wage** $w \geq 0$

27/ **Payoff of the worker**: $w - c(t, e)$, where $c(t, e)$ is the cost for a worker of ability t to acquire the level of education e

Profit of the employer: $y(t, e) - w$, where $y(t, e)$ is the productivity of a worker with ability t who obtained the level of education e

• **Perfect competition between employers** $\Rightarrow$ expected profits are zero $\Rightarrow$ the wage is equal to the expected productivity of the worker

➥ The payoff of the "representative" employer is, e.g., $-\big[y(t, e) - w\big]^2$

• We will see how the salary of the worker can increase with his level of education, even when it is common knowledge that the level of education has no impact on productivity. Indeed, we will assume from now on that

$$y(t, e) = y(t) = t$$

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• **Crucial assumption**: the marginal cost of education is decreasing with the worker's ability (**single-crossing**, **Spence-Mirrlees** property)

$$0 < \frac{\partial\, c(t^H, e)}{\partial e} < \frac{\partial\, c(t^L, e)}{\partial e} \quad \forall\, e \geq 0$$

for example $c(t, e) = e/t$

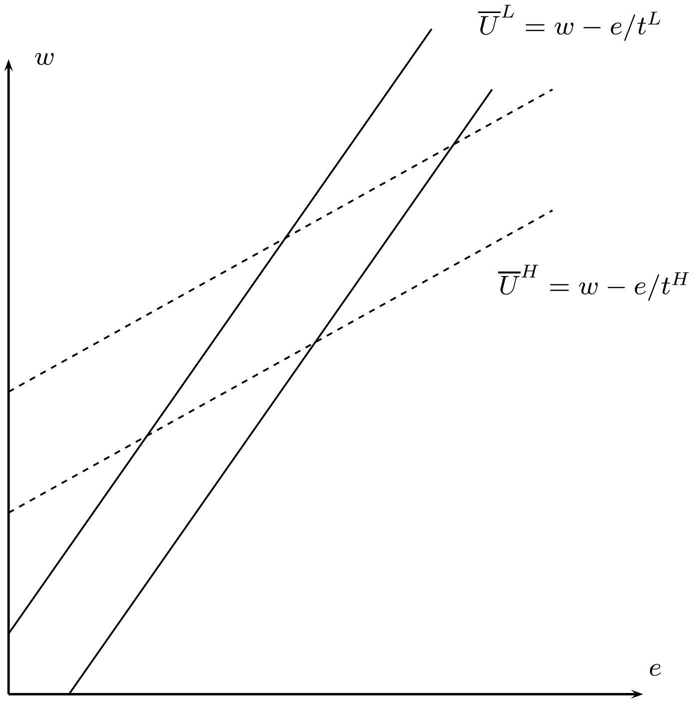

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Figure 1: Spence-Mirrlees ("single-crossing") property: the marginal cost of education decreases with the worker's ability

Remark If the worker's ability is common knowledge we have $w(e) = y(t, e)$

The worker would then choose e so as to maximize $w(e) - c(t, e)$

With our assumptions ($y(t, e) = t$) we would have $w(e) - c(t, e) = t - c(t, e)$ so the worker would choose $e = 0$

This "first best" solution (for the worker) is obviously not an equilibrium under asymmetric information since $e(t^L) = e(t^H) = 0$ does not reveal the worker's ability to the employer

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- **Sequential rationality of the employer**: for every $e \geq 0$,

$$\begin{aligned} w(e) &= \mathrm{E}_\mu\big[y(\cdot, e) \mid e\big] \\ &= \mu(t^H \mid e)\, y(t^H, e) + \mu(t^L \mid e)\, y(t^L, e) \\ &= \mu(t^H \mid e)\,(t^H - t^L) + t^L \end{aligned}$$

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- **Sequential rationality of the worker**: for every $t \in \{t^H, t^L\}$,

$$e(t) \in \arg\max_e \; w(e) - c(t, e) = \arg\max_e \; \mu(t^H \mid e)\,(t^H - t^L) - e/t$$

Pooling Equilibria

➥ The level of education does not depend on the worker's ability:

$$e(t^L) = e(t^H) = e_m$$

$$\Rightarrow \;\; \mu(t^H \mid e_m) = p \;\; \Rightarrow \;\; w(e_m) = p\,(t^H - t^L) + t^L$$

Worker t's payoff when he chooses e_m :

$$w(e_m) - c(t, e_m) = p\,(t^H - t^L) + t^L - e_m/t$$

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Worker t's payoff if he deviates to $e \neq e_m$:

$$w(e) - c(t, e) = \mu(t^H \mid e)\,(t^H - t^L) + t^L - e/t$$

where $\mu(t^H \mid e) \in [0, 1]$ is the off the equilibrium belief of the employer. It can be chosen arbitrarily since Bayes' rule does not apply

The worker does not deviate if

$$p\,(t^H - t^L) - e_m/t \geq \mu(t^H \mid e)\,(t^H - t^L) - e/t \;\; \forall\, t, \; \forall\, e \geq 0$$

The easiest way to satisfy this condition is to choose $\mu(t^H \mid e) = 0$ for all $e \neq e_m$

Hence, the worker does not deviate if

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$$\begin{aligned} & p\,(t^H - t^L) - e_m/t \geq -e/t \;\; \forall\, t, \; \forall\, e \geq 0 \\ \Leftrightarrow\;\; & p\,(t^H - t^L) - e_m/t \geq 0 \;\; \forall\, t \quad \Leftrightarrow \quad e_m \leq t\,p\,(t^H - t^L) \;\; \forall\, t \\ \Leftrightarrow\;\; & e_m \leq t^L\,p\,(t^H - t^L) \end{aligned}$$

Conclusion: Pooling PBE exist under the following conditions:
$e(t^L) = e(t^H) = e_m \leq t^L\,p\,(t^H - t^L)$

Those PBE can be supported with the following consistent beliefs

$$\mu(t^H \mid e) = \begin{cases} p & \text{if } e = e_m \\ 0 & \text{if } e \neq e_m \end{cases}$$

and the following sequentially rational strategy of the employer

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$$w(e) = \begin{cases} p\,(t^H - t^L) + t^L & \text{if } e = e_m \\ t^L & \text{if } e \neq e_m \end{cases}$$

✍ Show that there exist pooling Nash equilibria in which the worker chooses $e_m > t^L\,p\,(t^H - t^L)$ whatever his type. Explain why these Nash equilibria are not perfect Bayesian equilibria

Separating Equilibria

➥ The level of education depends on the ability of the worker:

$$e(t^L) = e^L \neq e(t^H) = e^H$$

$\Rightarrow$ $\mu(t^H \mid e^L) = 0$ and $\mu(t^H \mid e^H) = 1$

$\Rightarrow$ $w(e^L) = t^L$ and $w(e^H) = t^H$

Worker t^L's payoff when he chooses e^L:

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$$w(e^L) - c(t^L, e^L) = t^L - e^L/t^L$$

Worker t^H's payoff when he chooses e^H:

$$w(e^H) - c(t^H, e^H) = t^H - e^H/t^H$$

As before, the easiest way to support an equilibrium is to choose $\mu(t^H \mid e) = 0$ off the equilibrium path (for $e \notin \{e^L, e^H\}$)

$$\Rightarrow \; w(e) = \begin{cases} t^H & \text{if } e = e^H \\ t^L & \text{if } e \neq e^H \end{cases}$$

- Worker t^L does not deviate to $e \neq e^L$ if

$$t^L - e^L/t^L \geq w(e) - e/t^L \quad \forall\, e \neq e^L$$

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Since $w(0) = t^L$, this implies $e^L = 0$

The previous condition becomes $t^L \geq w(e) - e/t^L \quad \forall\, e \neq 0$

Since $w(e) = t^L$ for $e \neq e^H$ we get $t^L \geq w(e^H) - e^H/t^L$, i.e.,

$$\boxed{e^H \geq t^L\,(t^H - t^L)}$$

- Worker t^H does not deviate to $e \neq e^H$ if

$$\begin{aligned} & t^H - e^H/t^H \geq w(e) - e/t^H \quad \forall\, e \neq e^H \\ \Leftrightarrow\; & t^H - e^H/t^H \geq t^L - e/t^H \quad \forall\, e \neq e^H \\ \Leftrightarrow\; & t^H - e^H/t^H \geq t^L \end{aligned}$$

so

$$\boxed{e^H \leq t^H\,(t^H - t^L)}$$

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Conclusion: Separating PBE outcomes exist when $e(t^L) = 0$ and

$$t^L\,(t^H - t^L) \leq e(t^H) \leq t^H\,(t^H - t^L)$$

Those equilibria can be supported with the following consistent beliefs

$$\mu(t^H \mid e) = \begin{cases} 1 & \text{if } e = e^H \\ 0 & \text{if } e \neq e^H \end{cases}$$

and the following sequentially rational strategy of the employer

$$w(e) = \begin{cases} t^H & \text{if } e = e^H \\ t^L & \text{if } e \neq e^H \end{cases}$$

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Remark An more intuitive belief system, which is also consistent and support these equilibria, is $\mu(t^H \mid e) = \begin{cases} 1 & \text{if } e \geq e^H \\ 0 & \text{if } e < e^H \end{cases}$ so $w(e) = \begin{cases} t^H & \text{if } e \geq e^H \\ t^L & \text{if } e < e^H \end{cases}$

Remark Some stronger equilibrium refinements, based on forward induction (for example, the intuitive criterion of Cho and Kreps, 1987) allow to select as a unique equilibrium the most efficient separating equilibrium: $e(t^L) = 0$,
$e(t^H) = t^L\,(t^H - t^L)$

Idea of the refinement: If e is a strictly dominated action for the sender of type t, but not for the sender of type t', then $\mu(t \mid e) = 0$

Other applications:

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- *Advertising* (Milgrom and Roberts, 1986): A firm selling a high quality product can signal this quality with expensive advertising if a firm with a low quality product is not able to cover these advertising costs given its future profits
- *Insurance* (Rothschild and Stiglitz, 1976; Wilson, 1977): A risk-averse driver will purchase a lower cost, partial insurance contract, leaving the riskier driver to pay a high rate for full insurance

- *Bargaining*: The magnitude of the offer of a firm to a union may reveal its profitability if firms with low profits are better able to make low wage offers (because the threat of a strike is less costly to a firm with low profits)
- *Evolutionary biology* (Zahavi, 1975; Grafen, 1990: *handicap principle*): a peacock's tail may be a signal used by prospective mates in order to estimate the individual's overall condition and/or genetic quality. Indeed, only the strongest individuals should be able to survive to predators with such an handicap. The same principle can explain why gazelles jump up and down when they see a lion, . . .

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