Dynamic Games of Incomplete Information

Equilibrium Refinement and Signaling Games

Outline

(November 20, 2007)

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- Application: Spence's (1973) Model of Education

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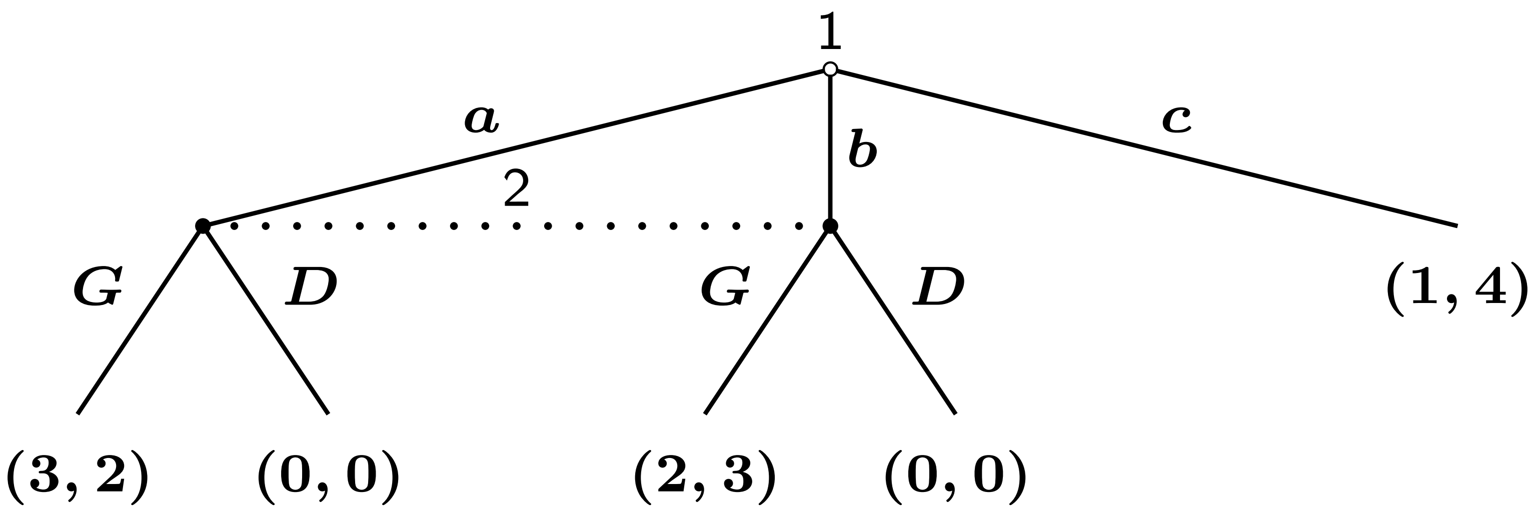

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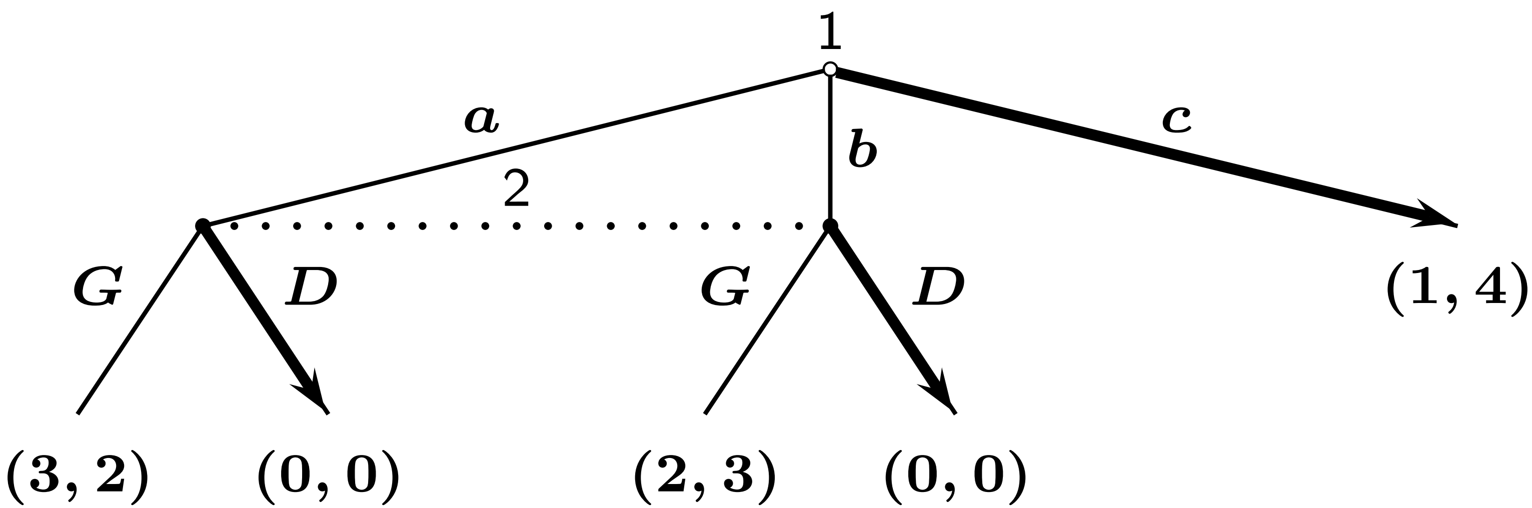

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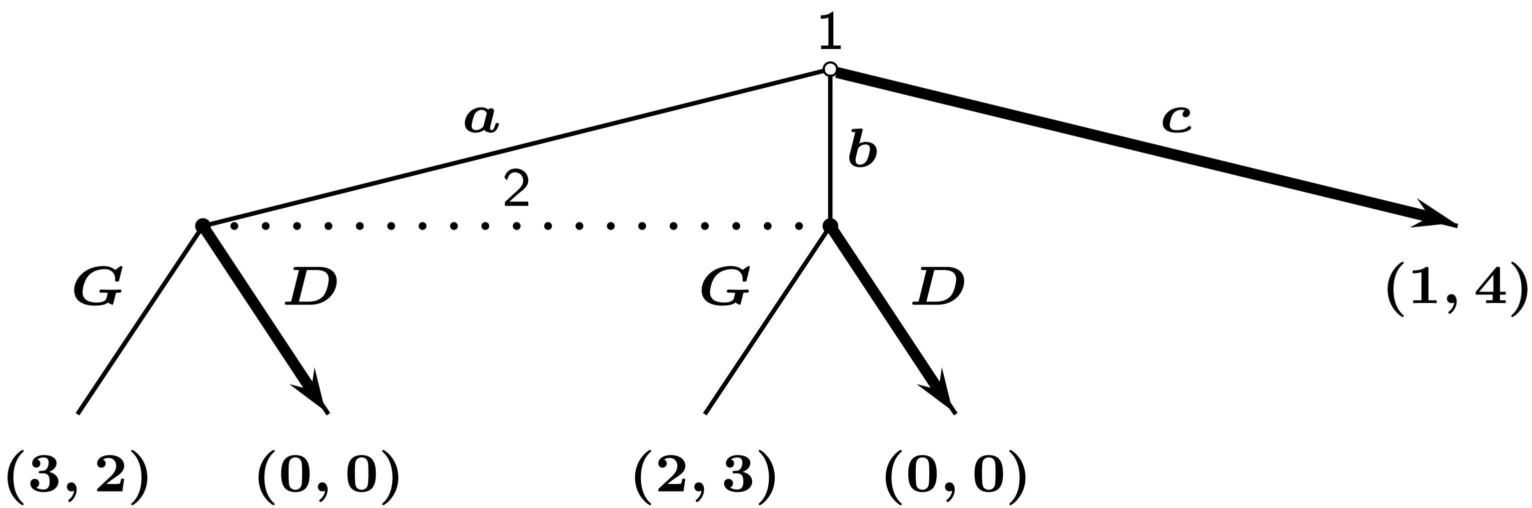

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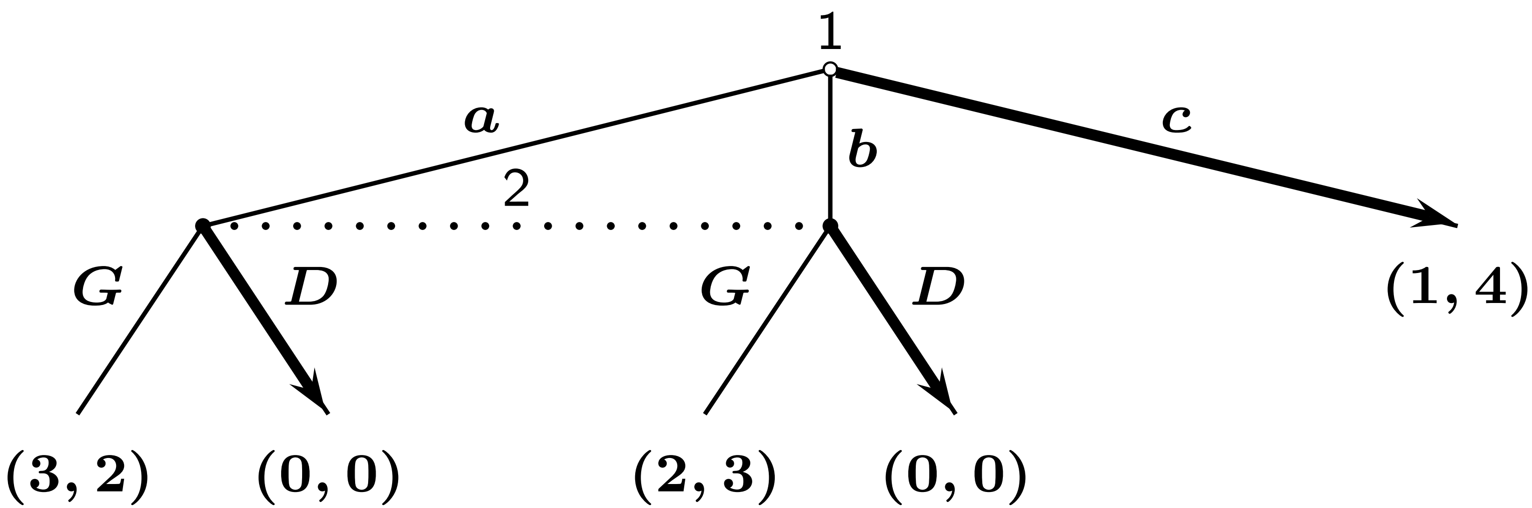

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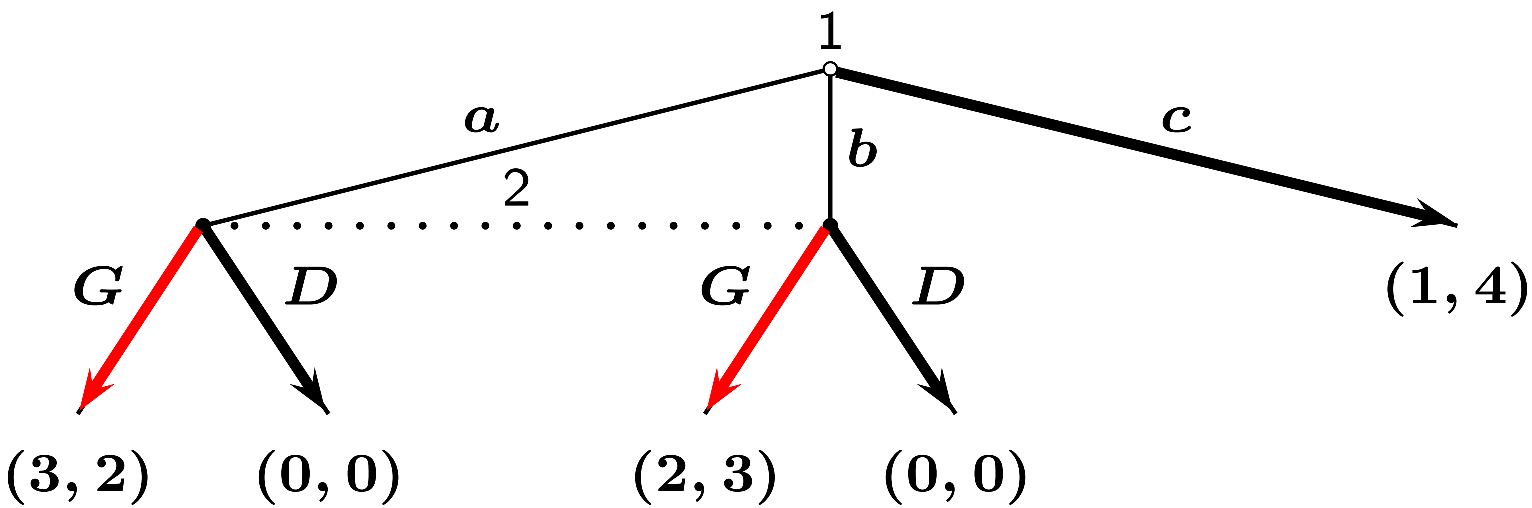

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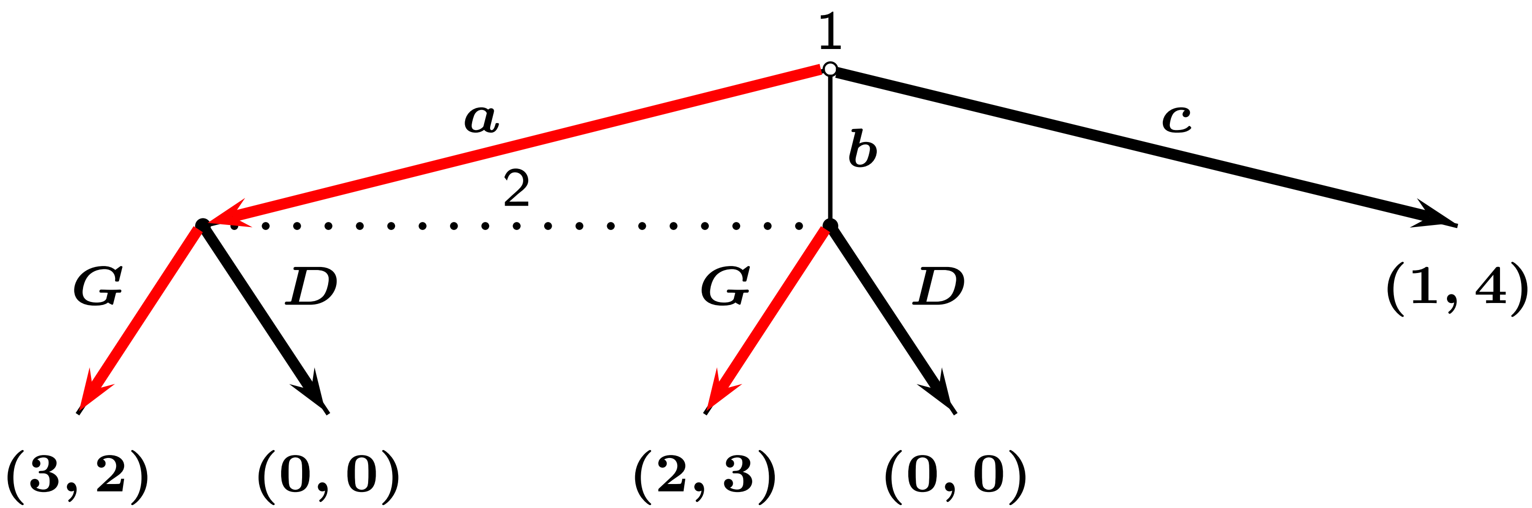

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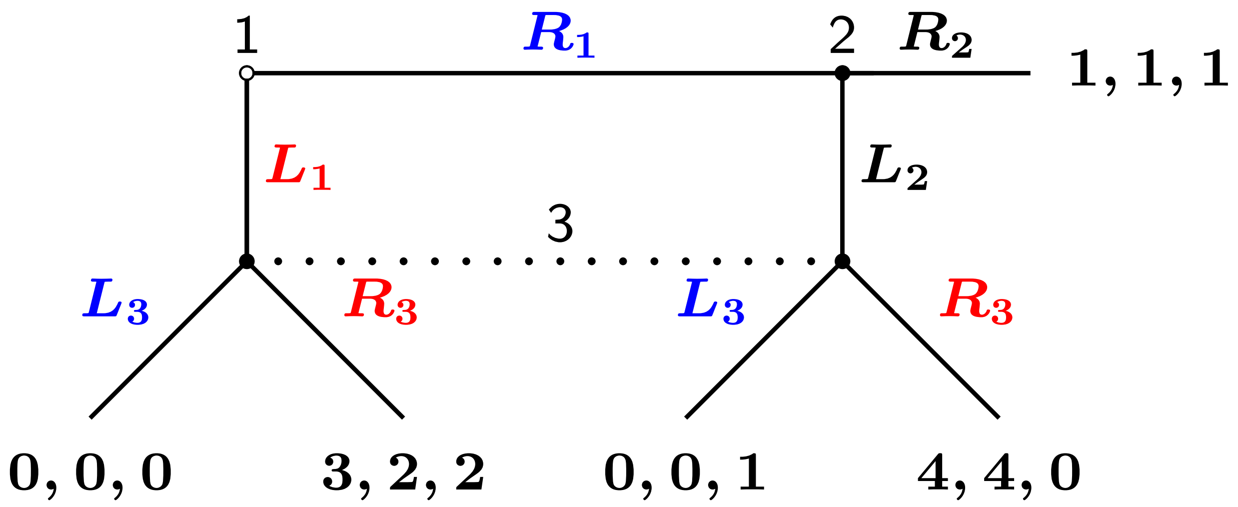

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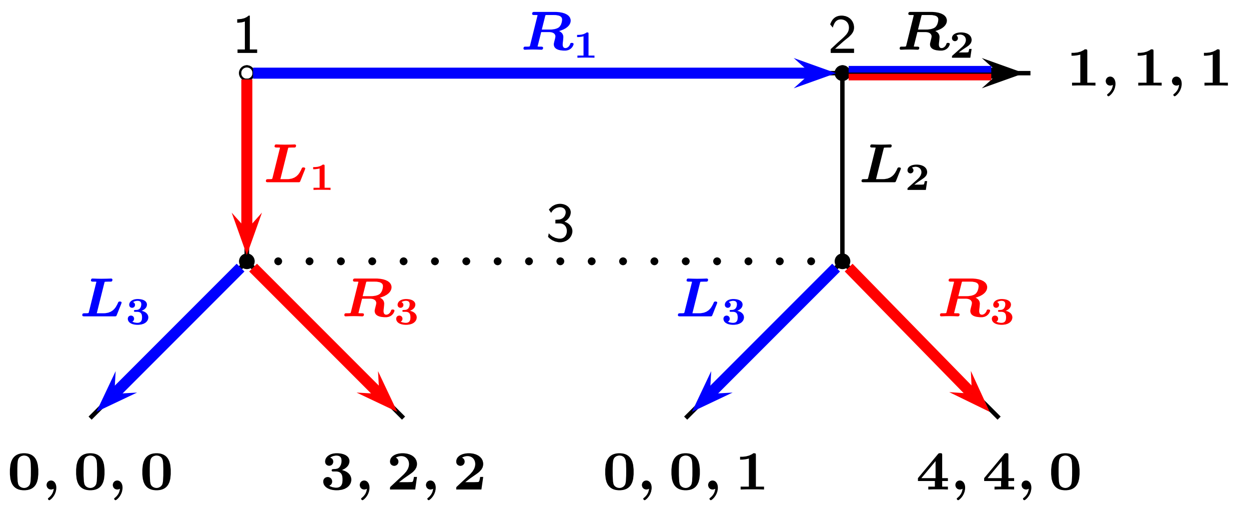

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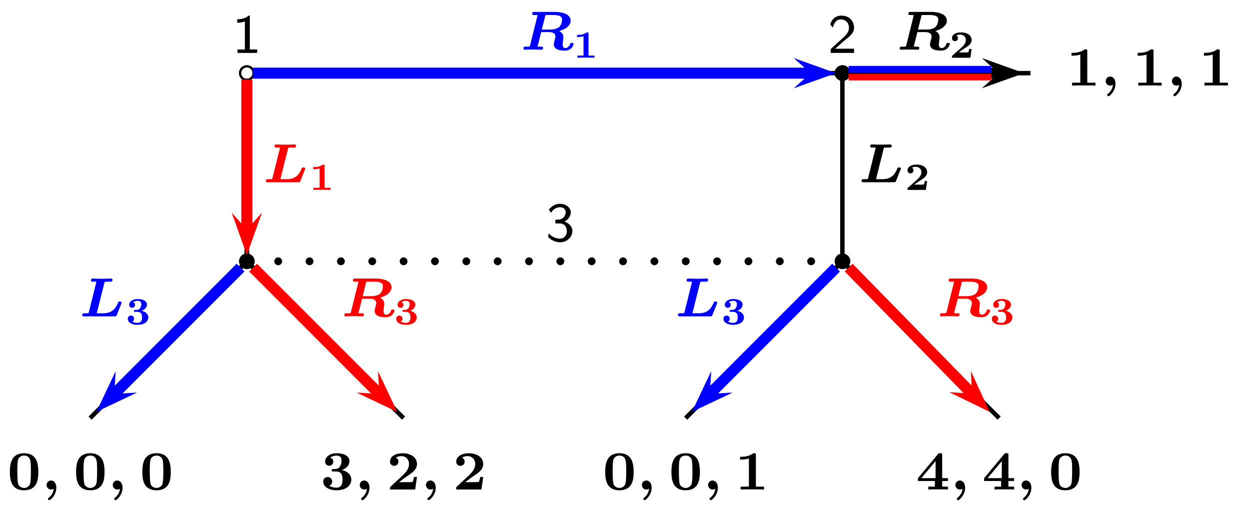

2 pure strategy (SP)NE: (R_1, R_2, L_3) and (L_1, R_2, R_3)

But in (L_1, R_2, R_3) the action R_2 of player 2 is not sequentially rational given that player 3 plays R_3 $(4 > 1)$

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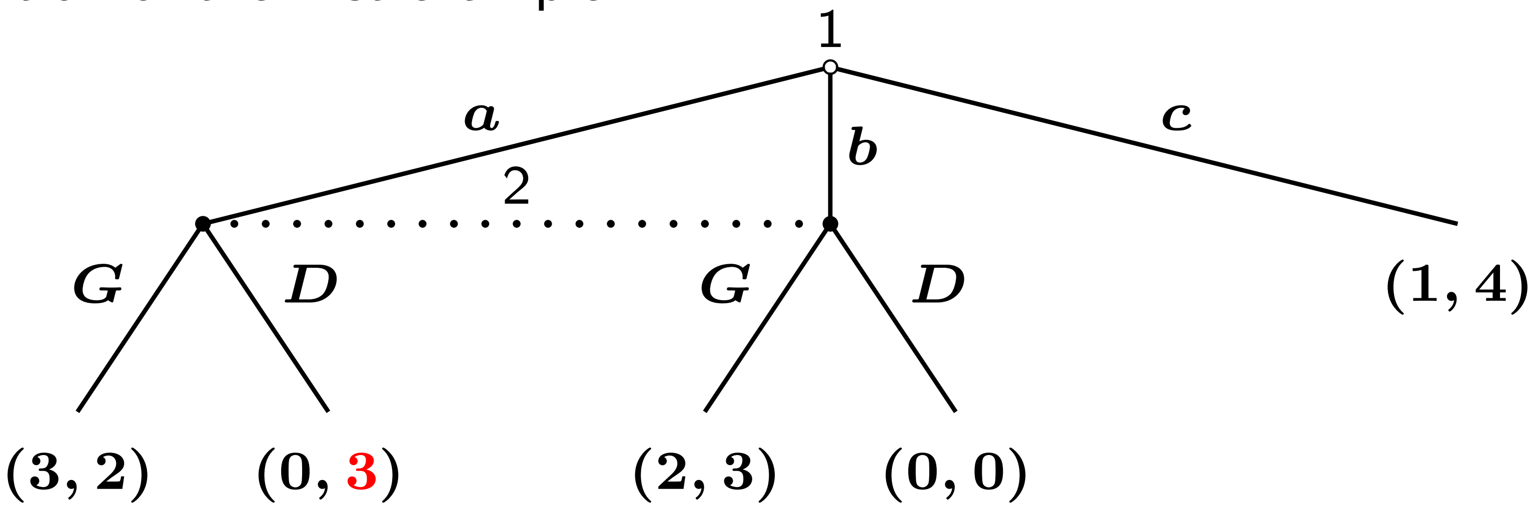

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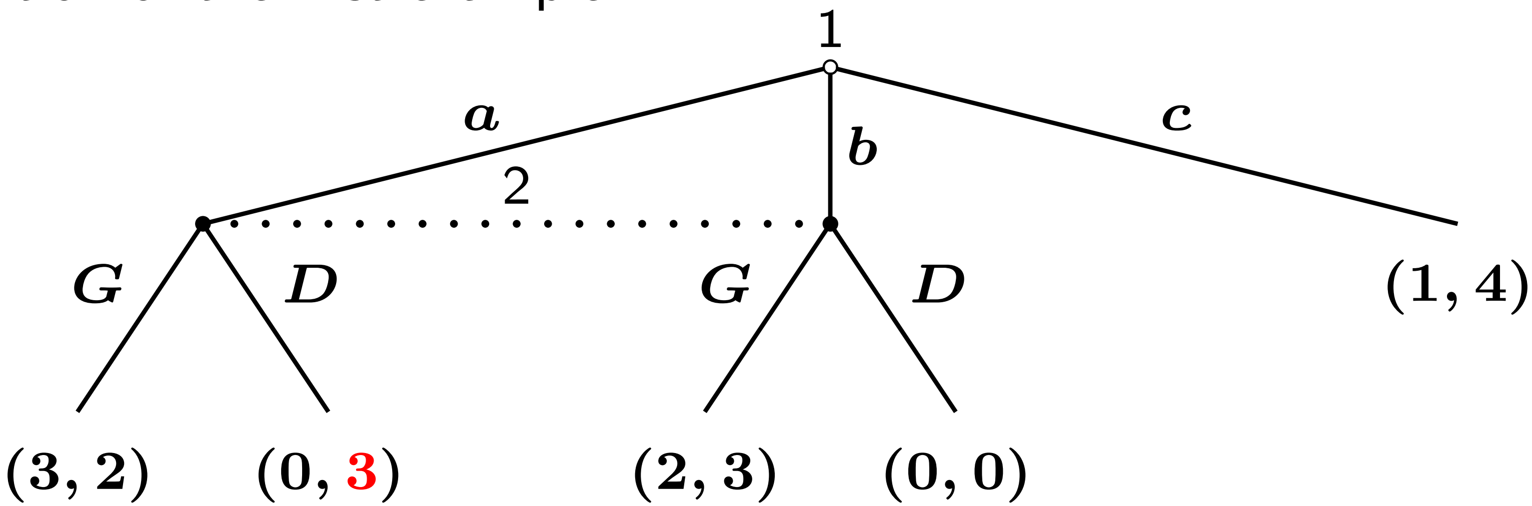

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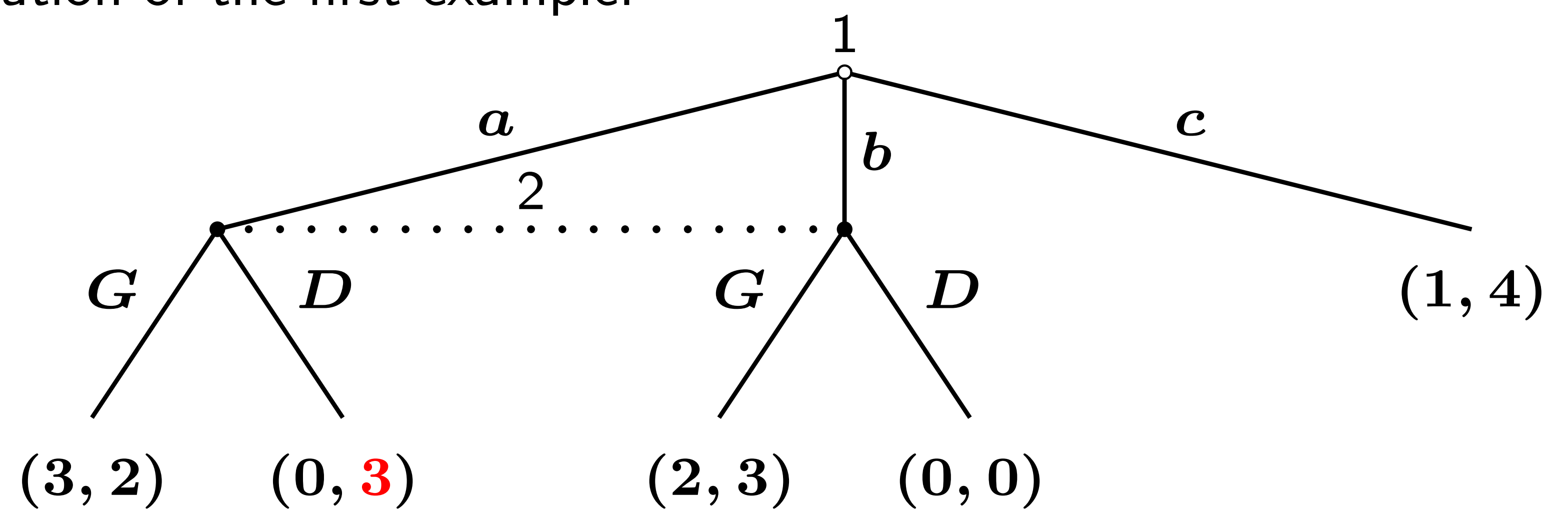

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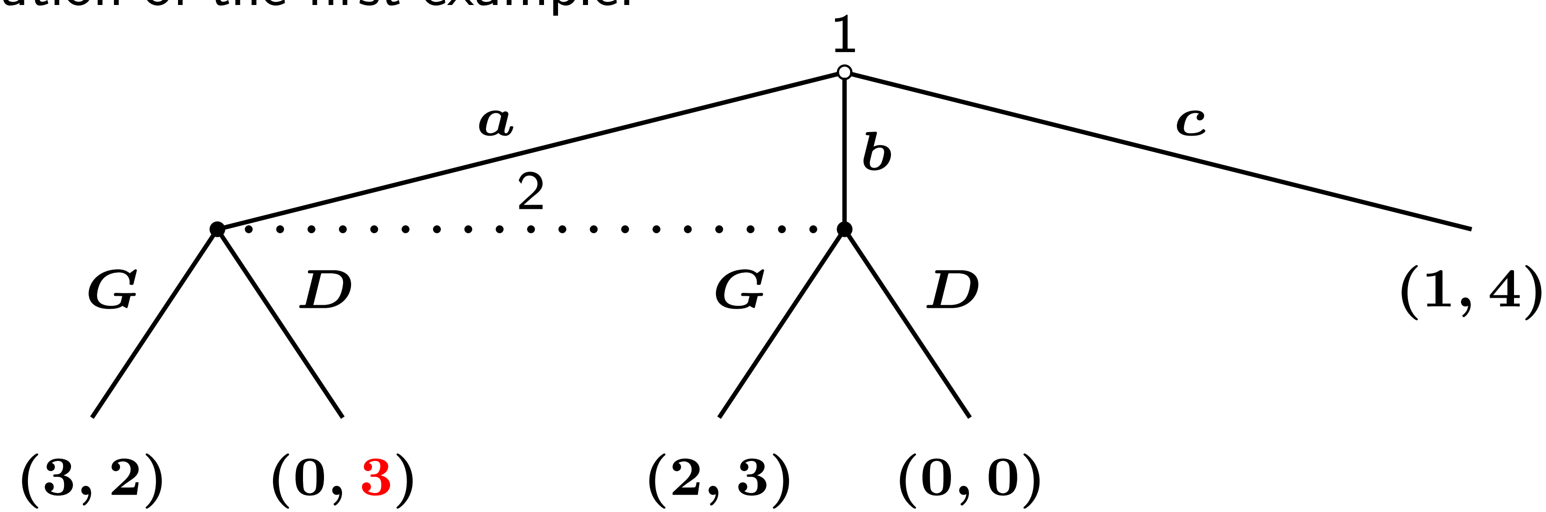

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➠ The solution concept is not only characterized by a strategy profile but also by a **belief system**

Belief System

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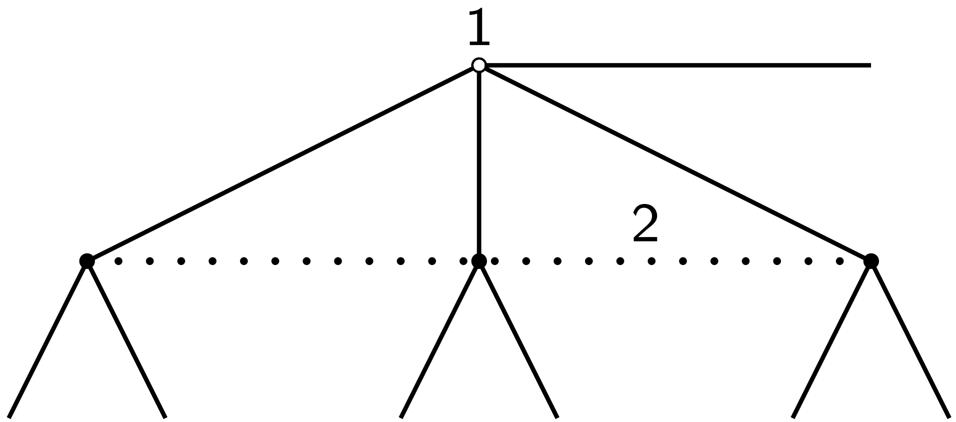

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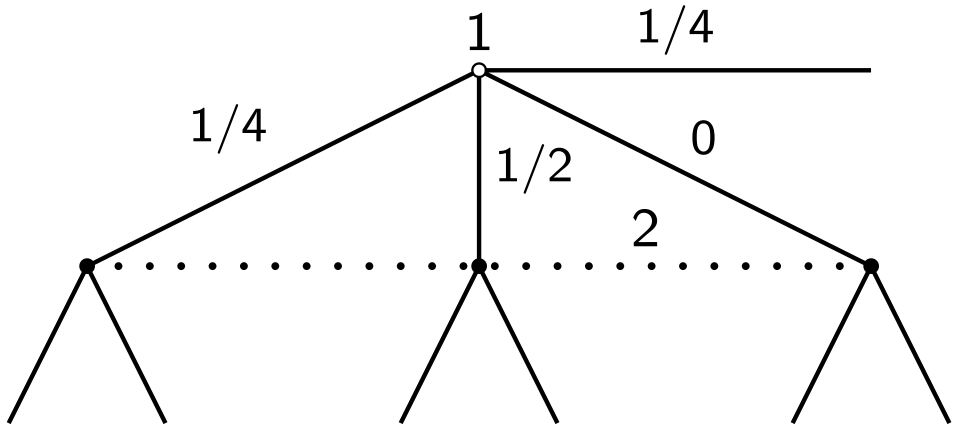

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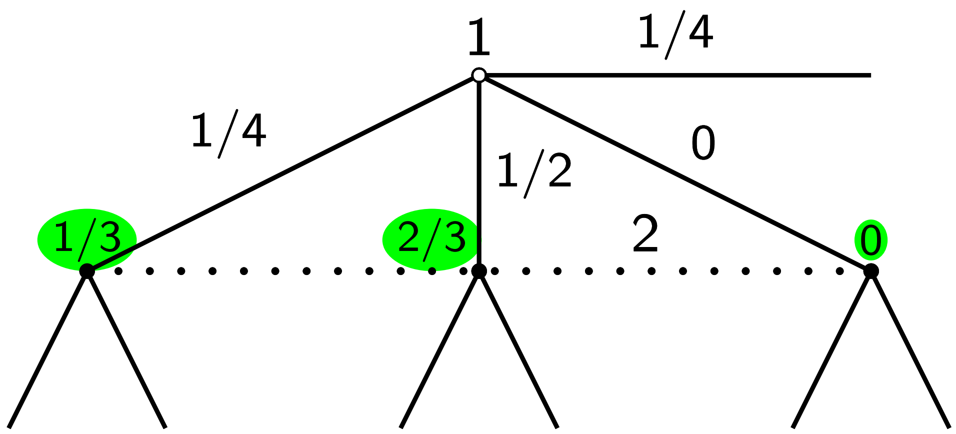
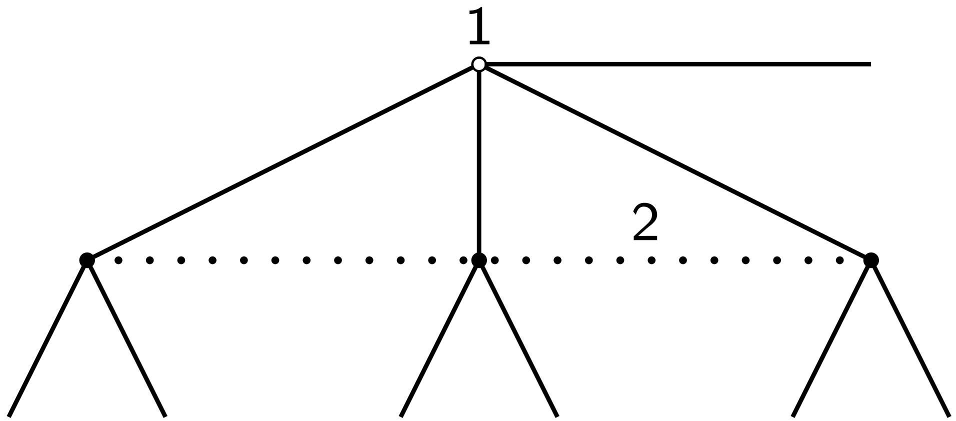


➥ Bayes' rule can be applied: $\boldsymbol{\mu_2 = (\frac{1}{3}, \frac{2}{3}, 0)}$


1
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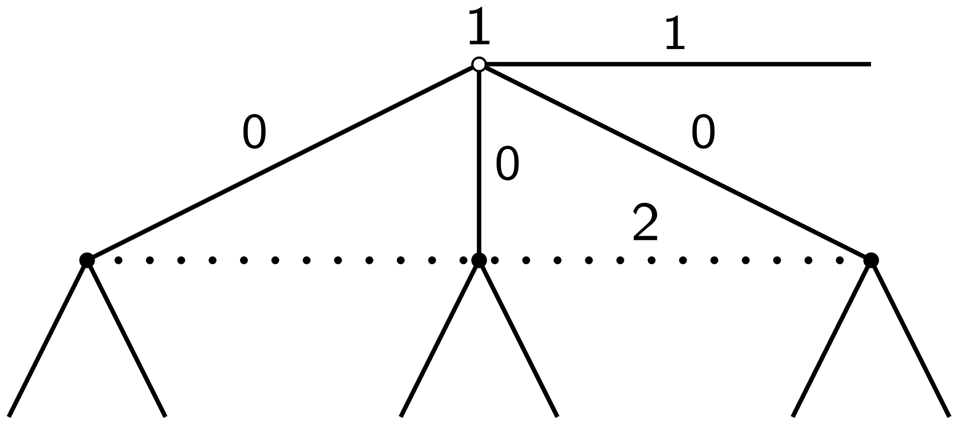
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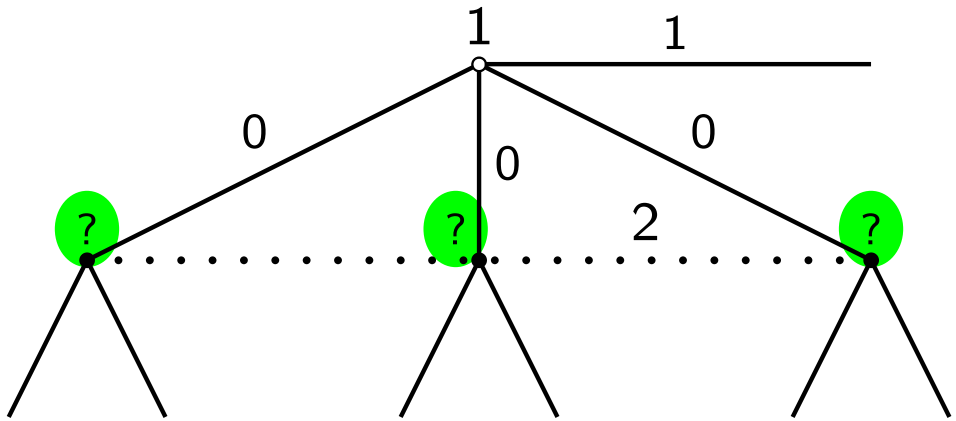

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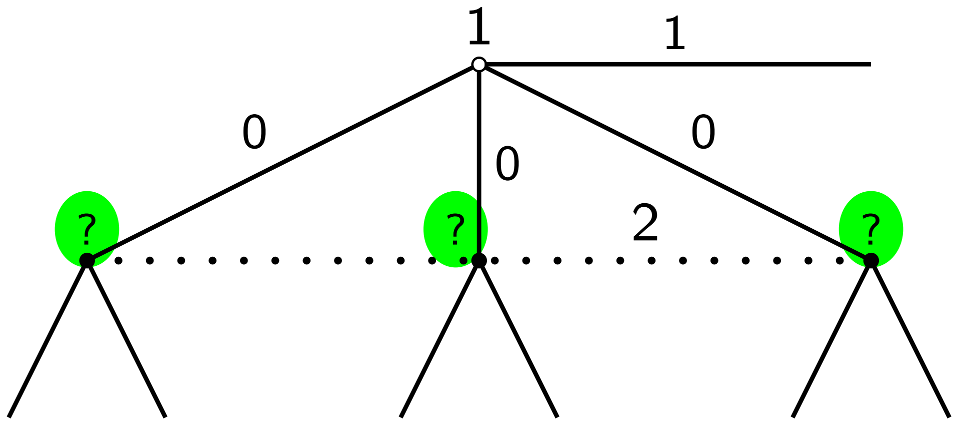

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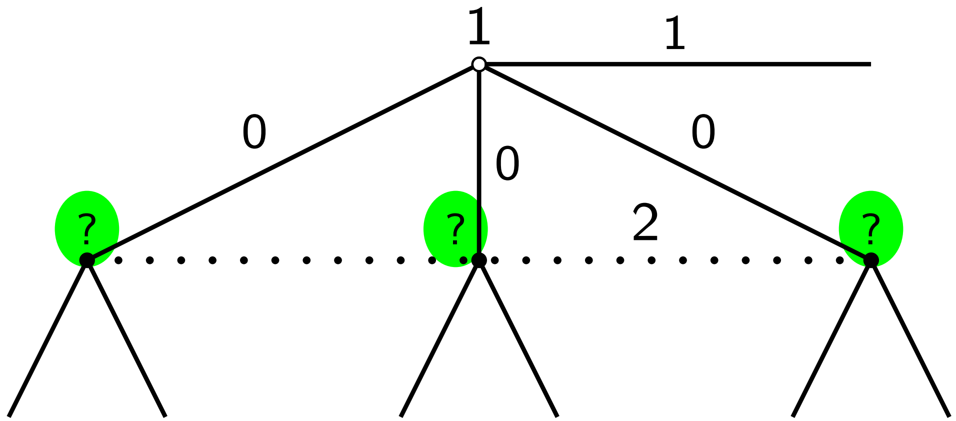
Belief system: collection of probability distributions on decision nodes, one distribution for each information set



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☞ trivial in perfect information games (probability 1 at every node)

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- **Sequential Rationality.** For every player $\boldsymbol{i}$ and every information set of player $\boldsymbol{i}$, the local strategy of player $\boldsymbol{i}$ at this information set maximizes his expected utility given his belief at this information set and the strategies of the other players

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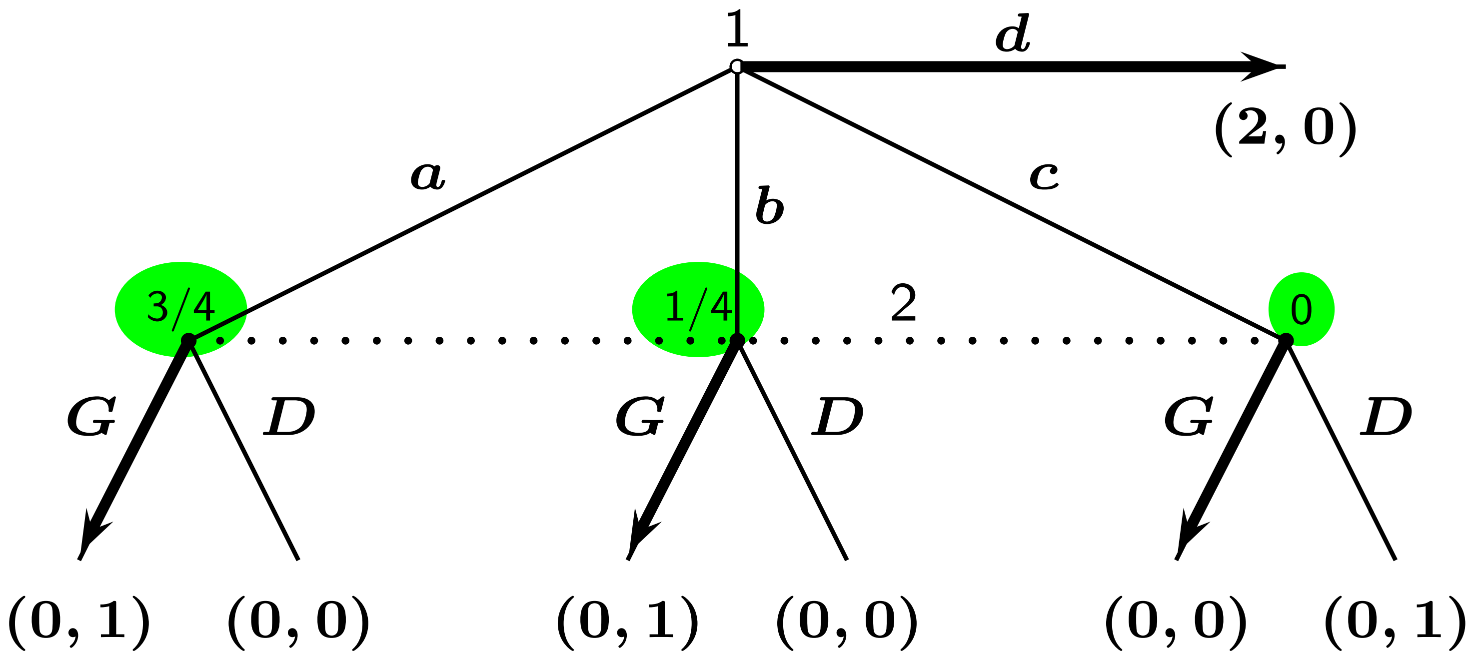
- **Sequential Rationality.** For every player i and every information set of player i, the local strategy of player i at this information set maximizes his expected utility given his belief at this information set and the strategies of the other players

- **Weak Belief Consistency.** In every subgame (along and off the equilibrium path), beliefs are computed by Bayes' rule according to $\boldsymbol{\sigma}$ when it is possible. When Bayes' rule cannot be applied, beliefs can be chosen arbitrarily

Example. (d, G) is a perfect Bayesian equilibrium (PBE)



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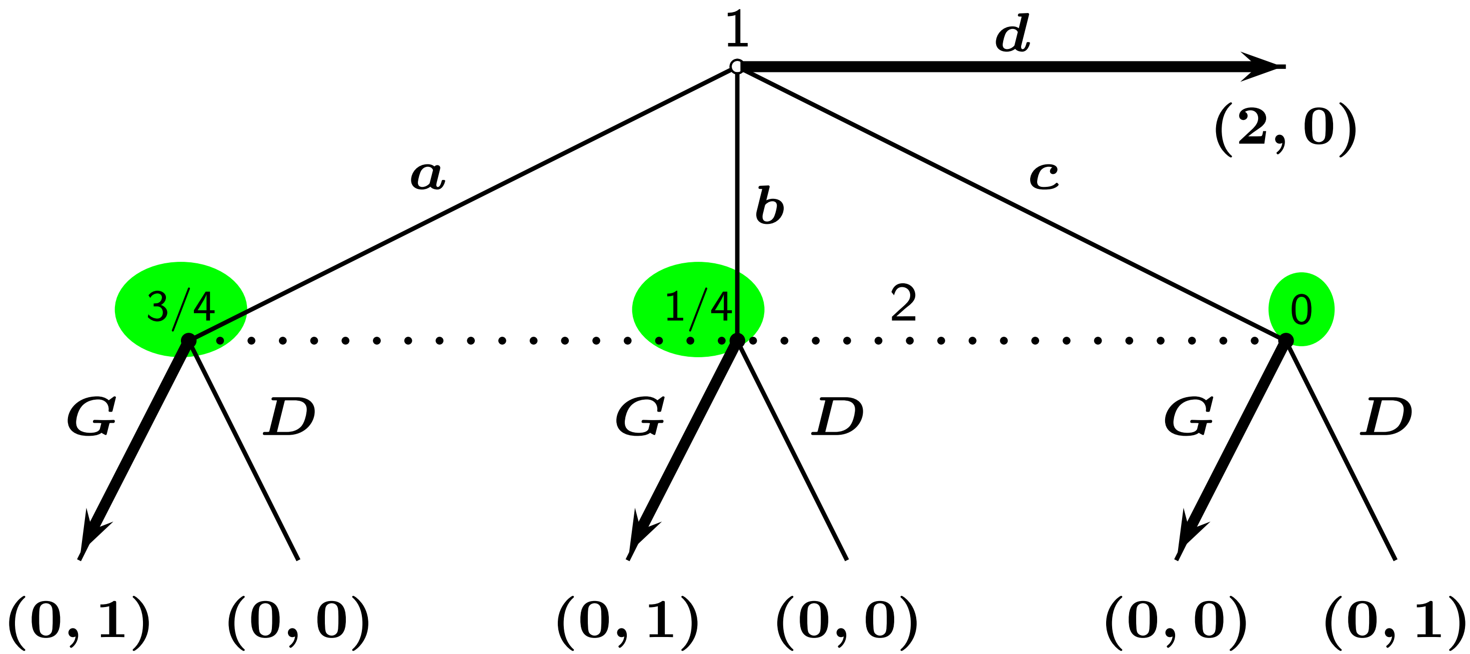

Remark. Many other belief systems are possible $((1, 0, 0)$, $(0, 1, 0)$, $(1/3, 1/3, 1/3)$, $\ldots)$

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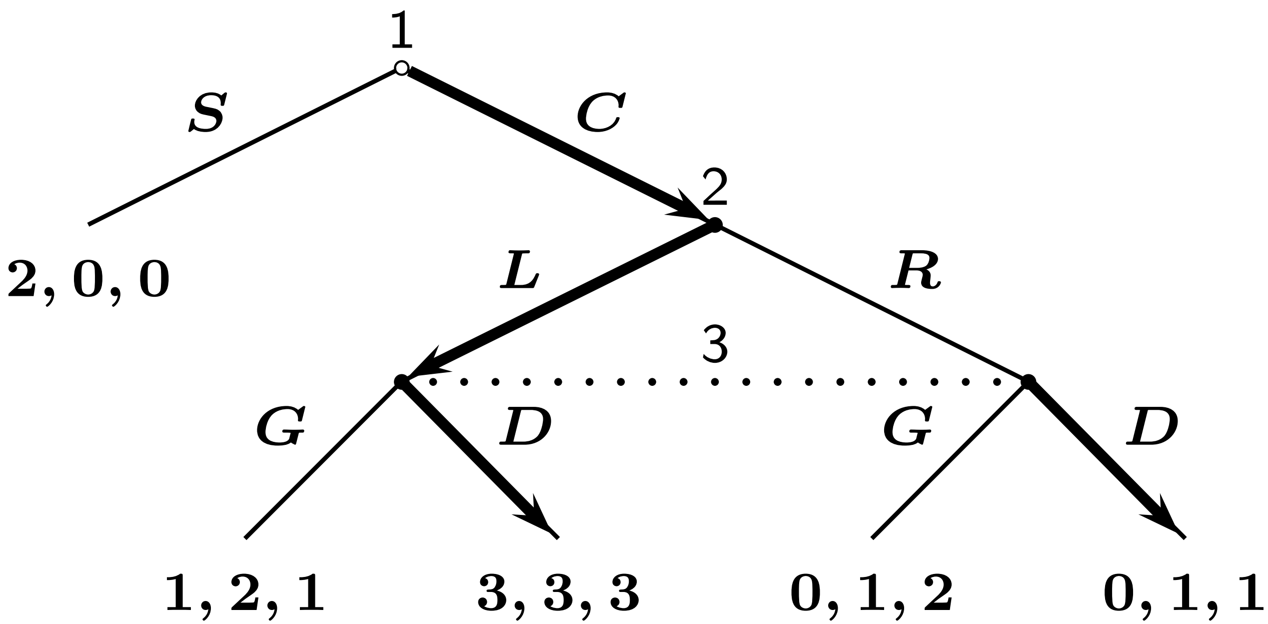

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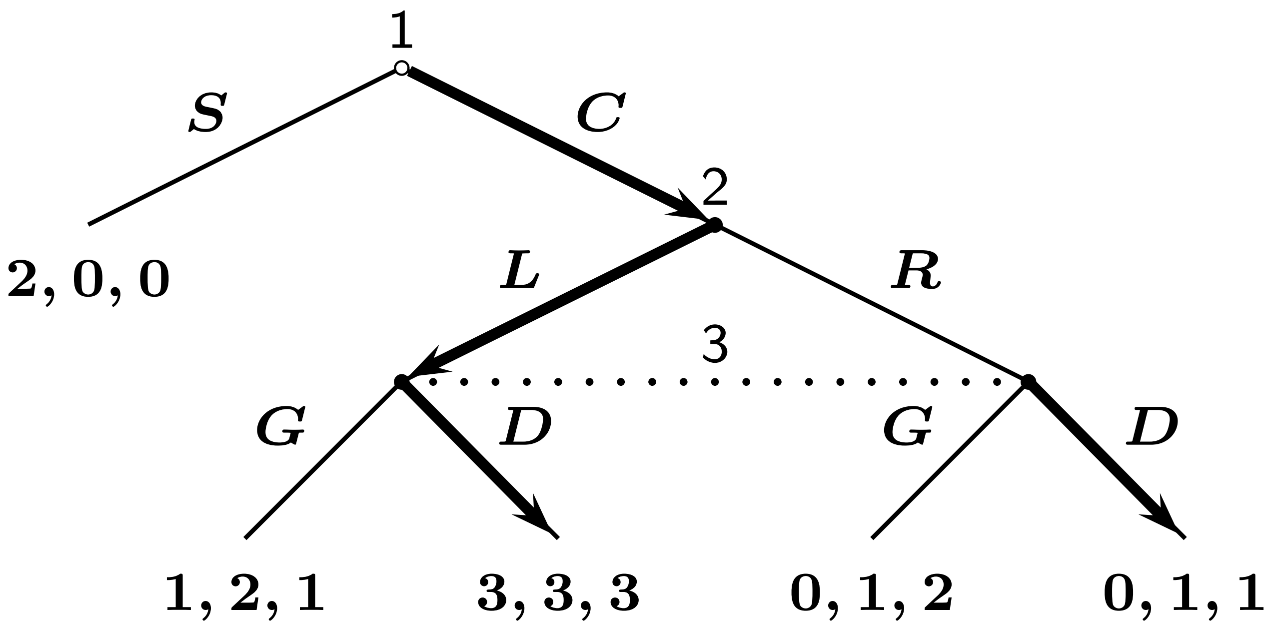

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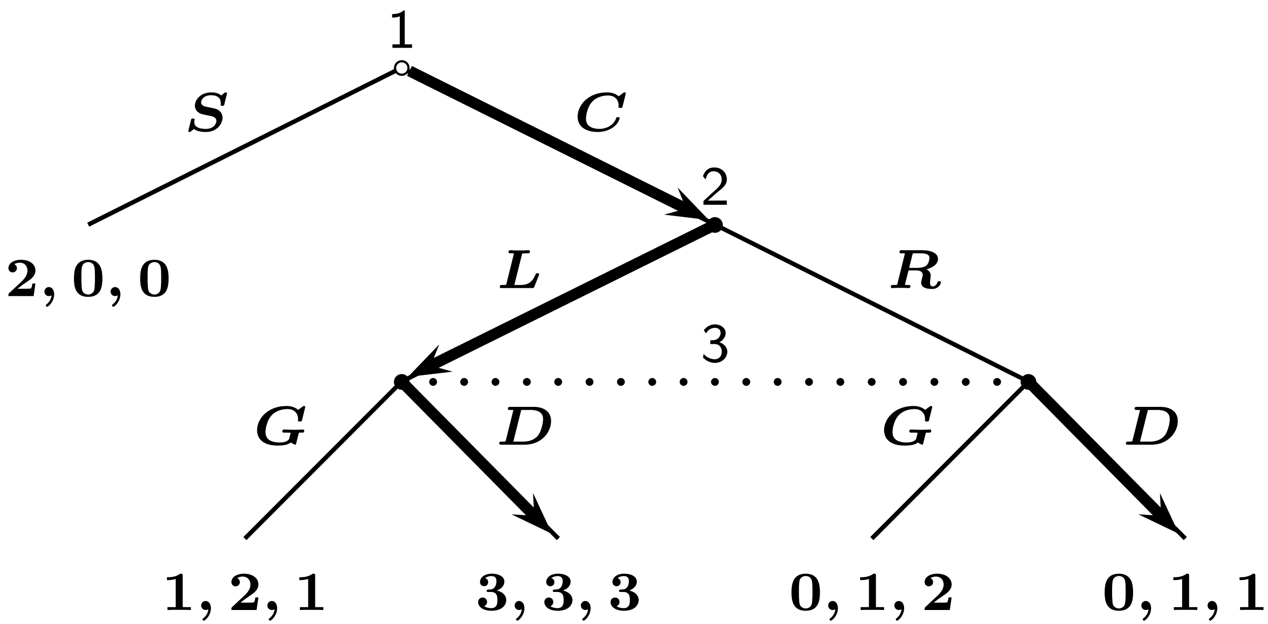

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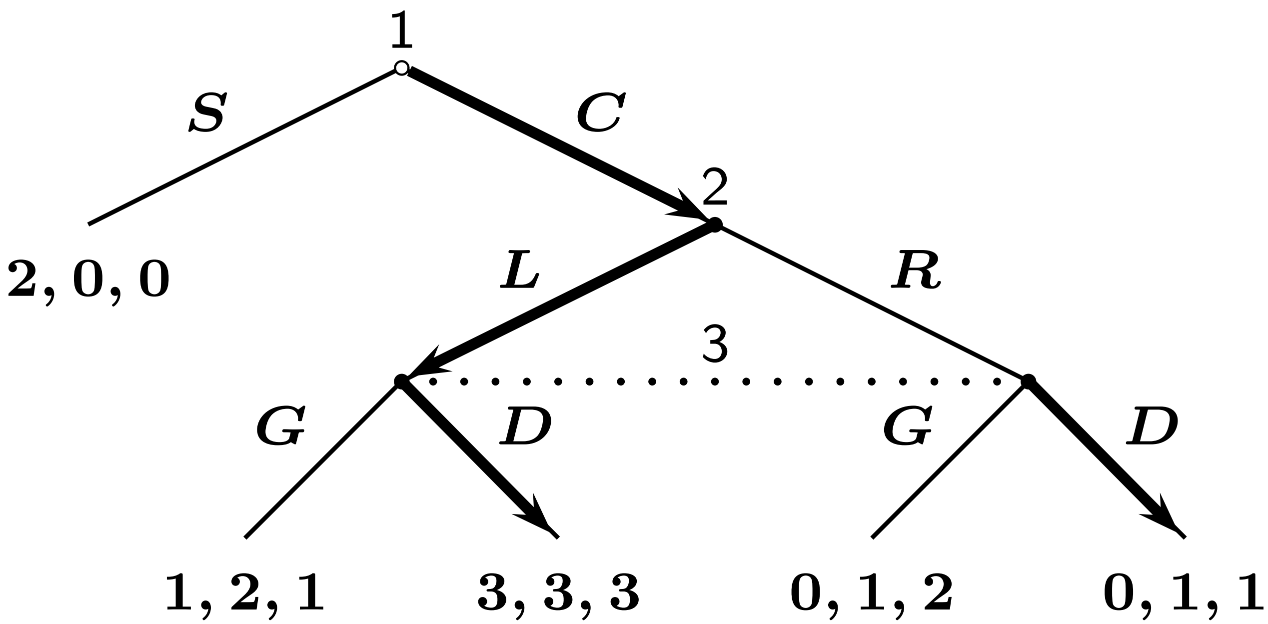

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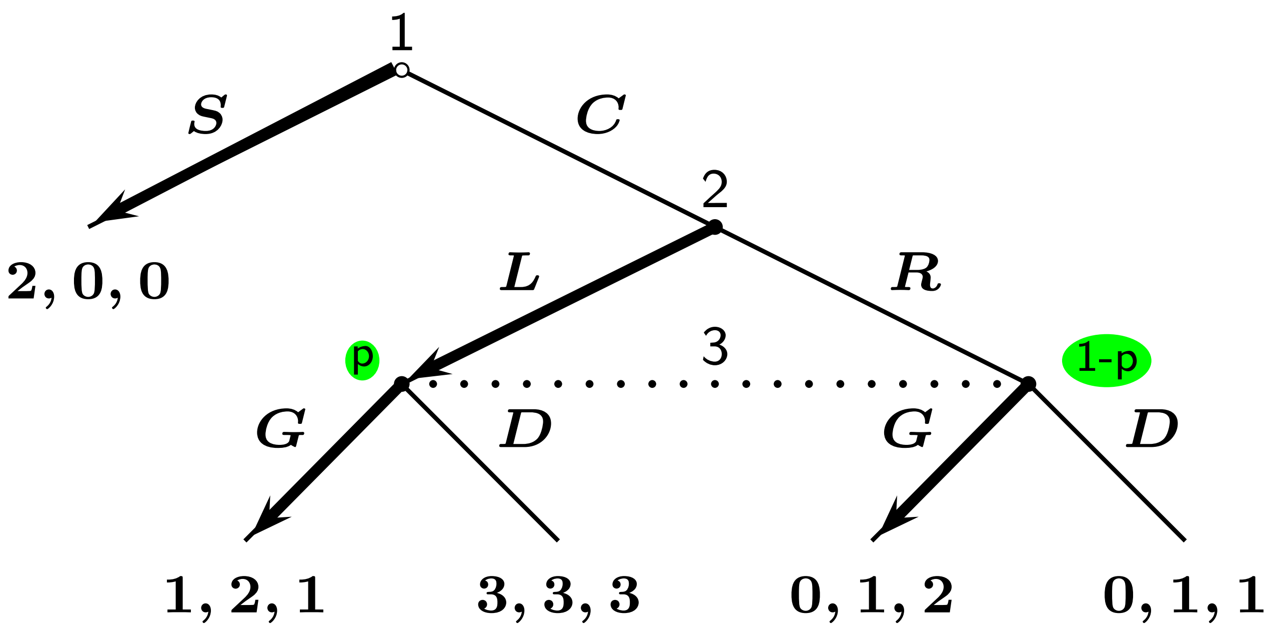
Sequential rationality is satisfied

Next, consider the Nash equilibrium (S, L, G) (which is not subgame perfect)



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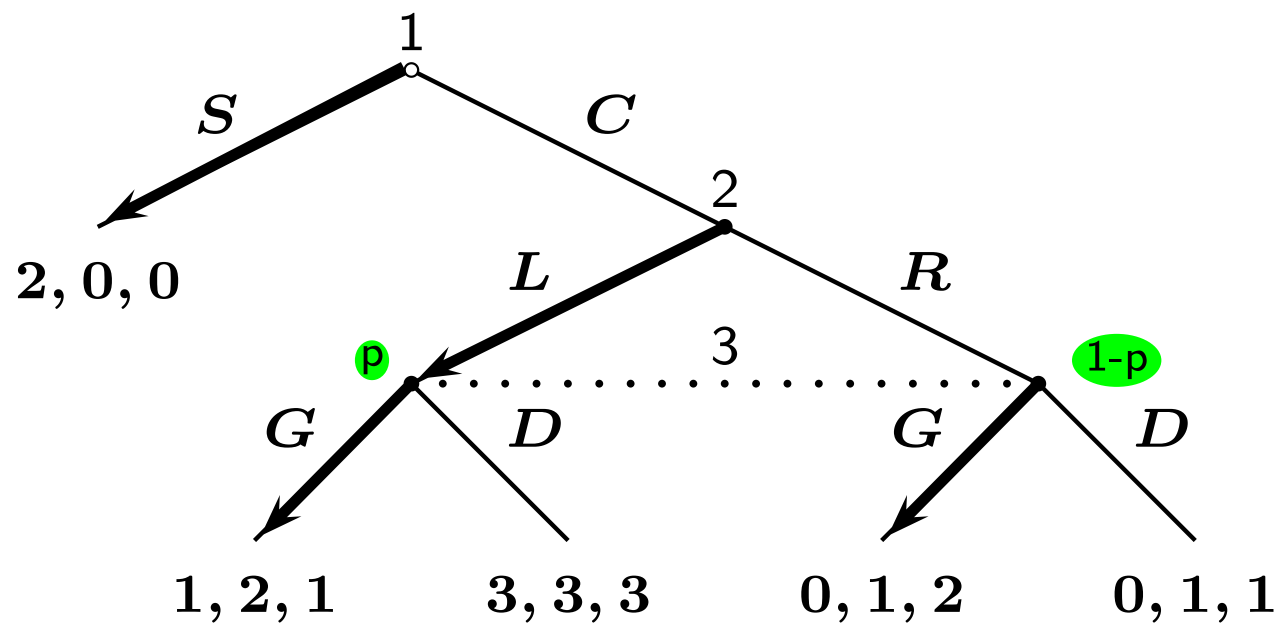

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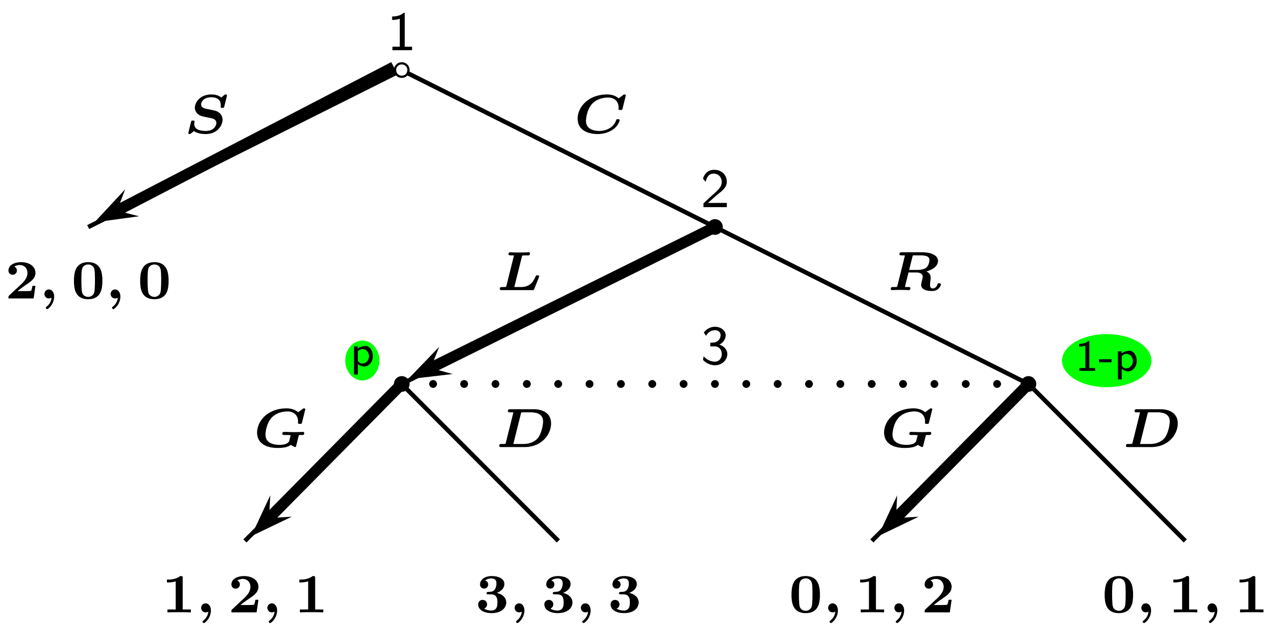

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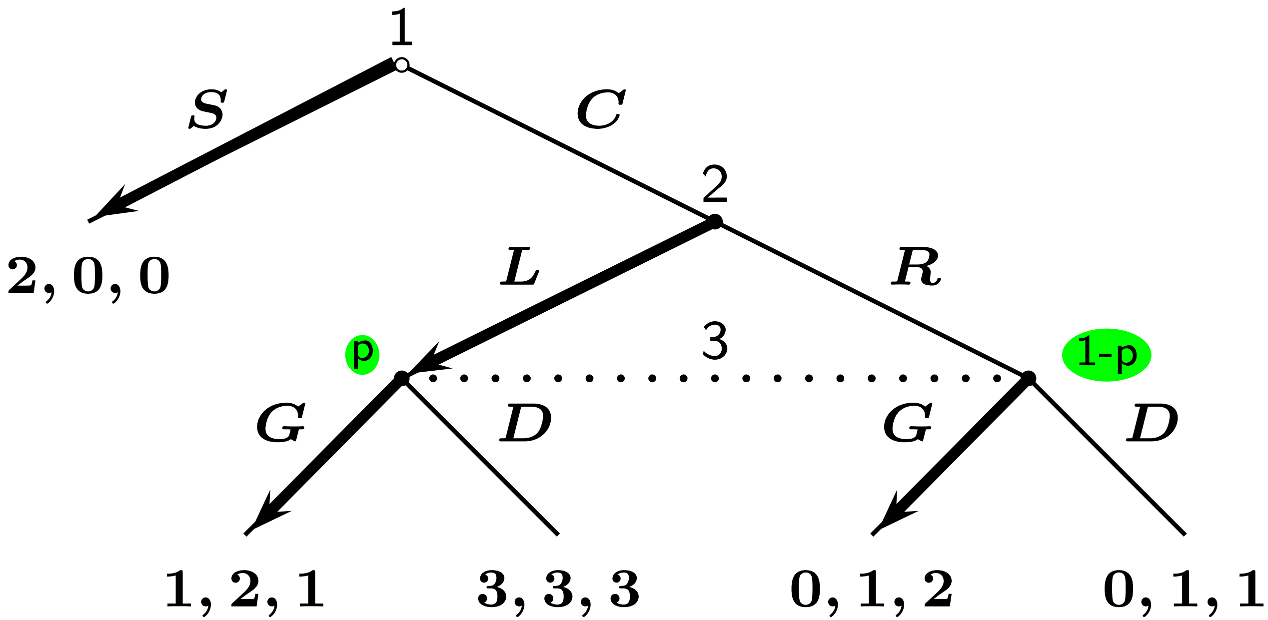

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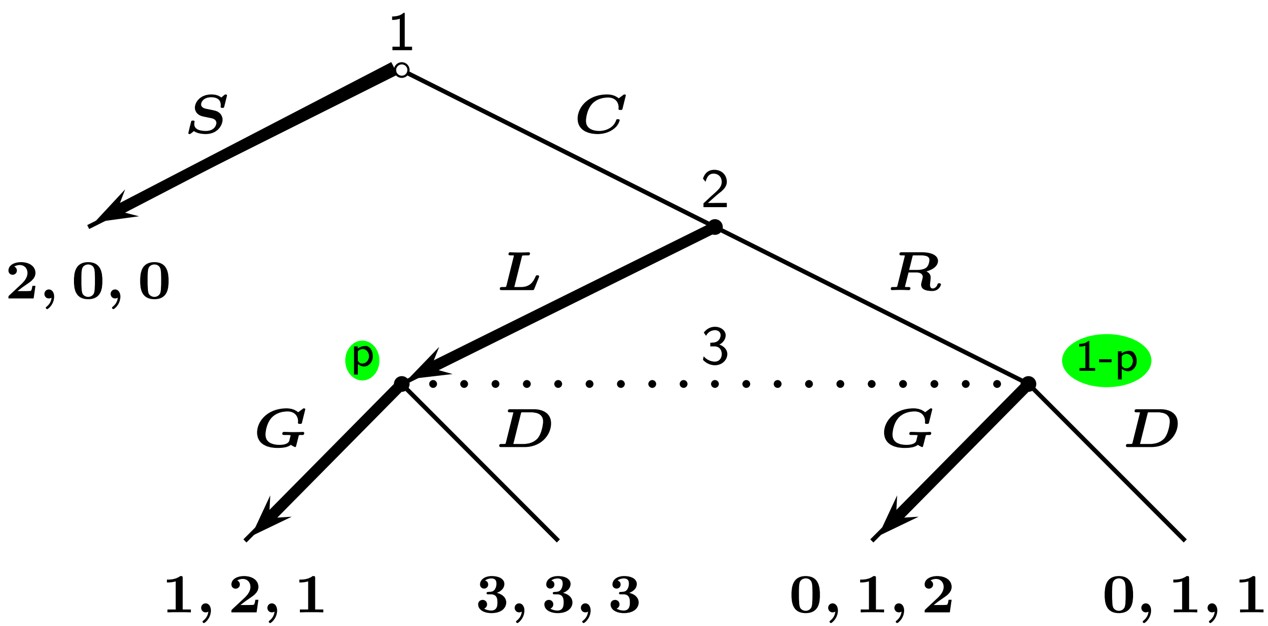

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But μ_3 is not weakly consistent because in the strict subgame (off the equilibrium path) Bayes' rule implies $p = 1$

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Strictly positive strategy of player i : $\sigma_{h_i}(a_i) > 0$ for every action available at information set h_i of player i, $a_i \in A(h_i)$, and for every information set of player i, $h_i \in H_i$

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Strong sequential equilibrium (SE): Sequential rationality + **strong** belief consistency

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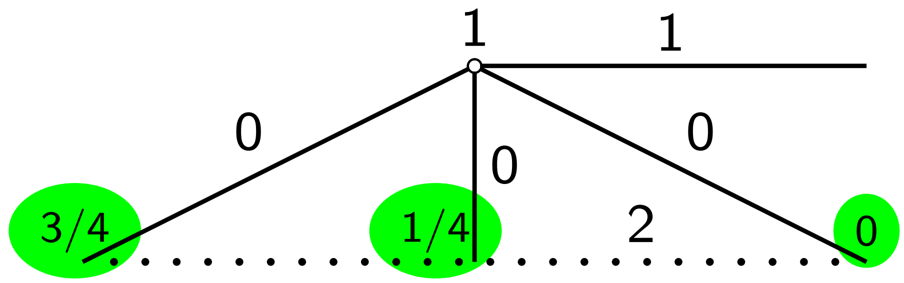

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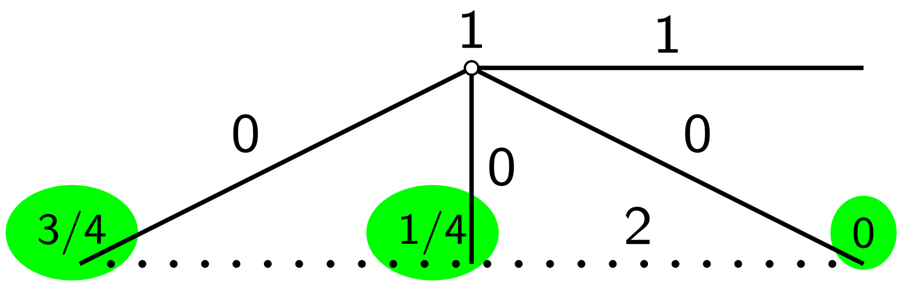

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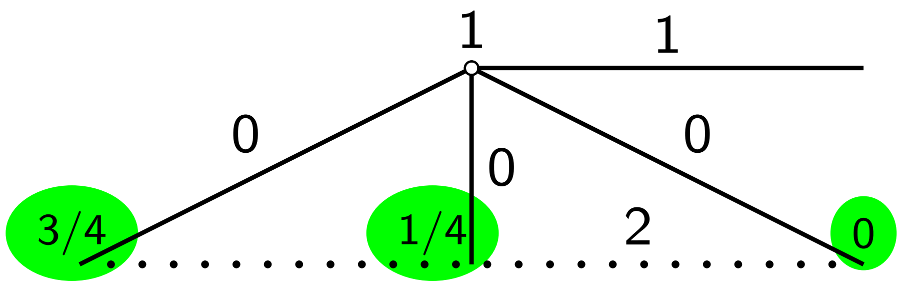

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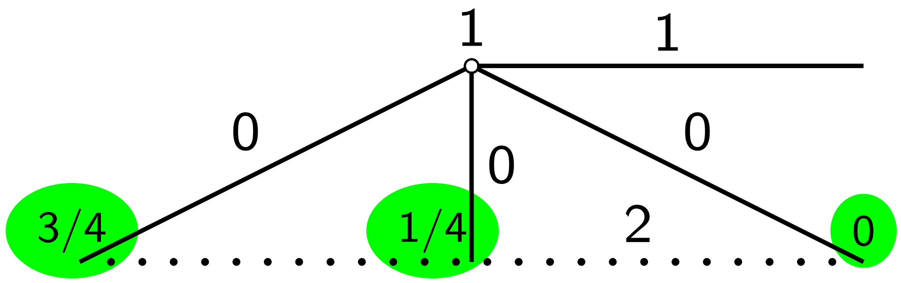

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Remark Strong belief consistency requires finite action sets and state spaces (except in the last decision nodes of the game tree)

Example of a PBE which is not a (strong) sequential equilibrium

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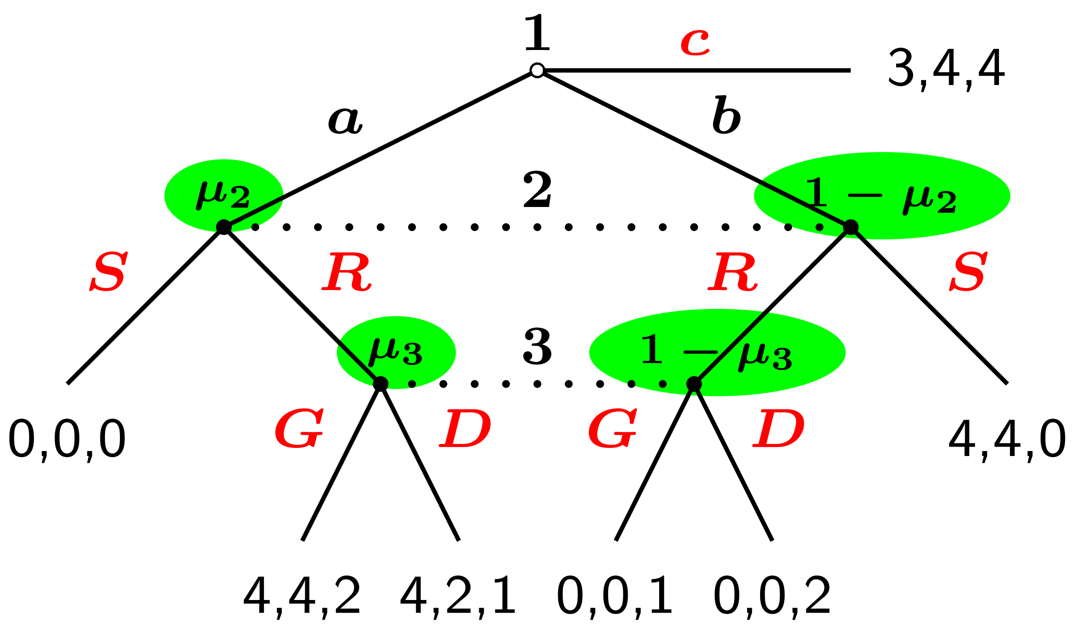

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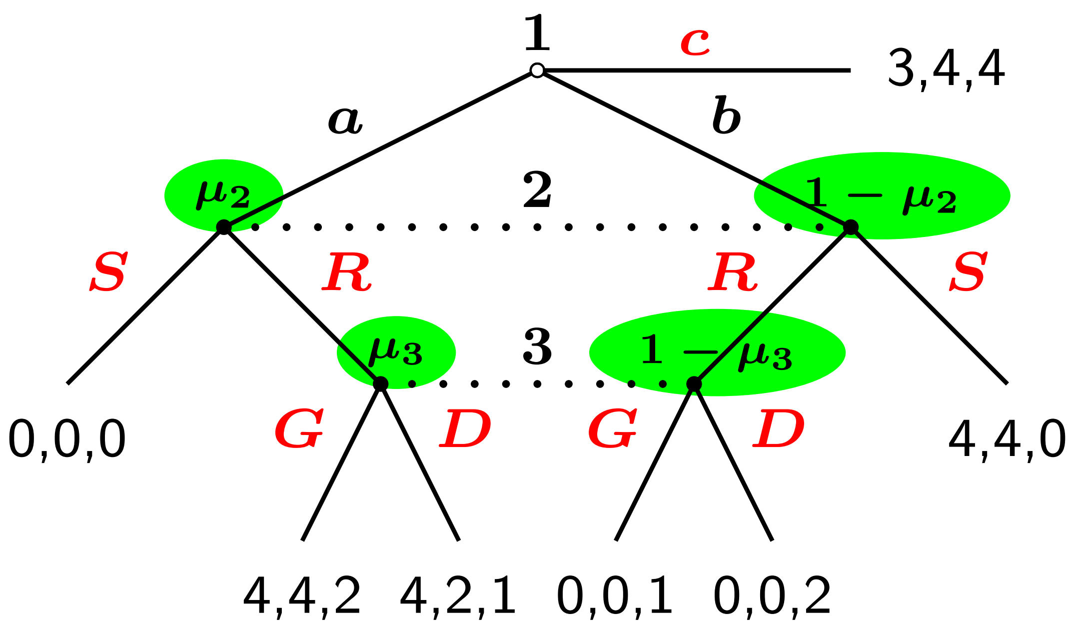

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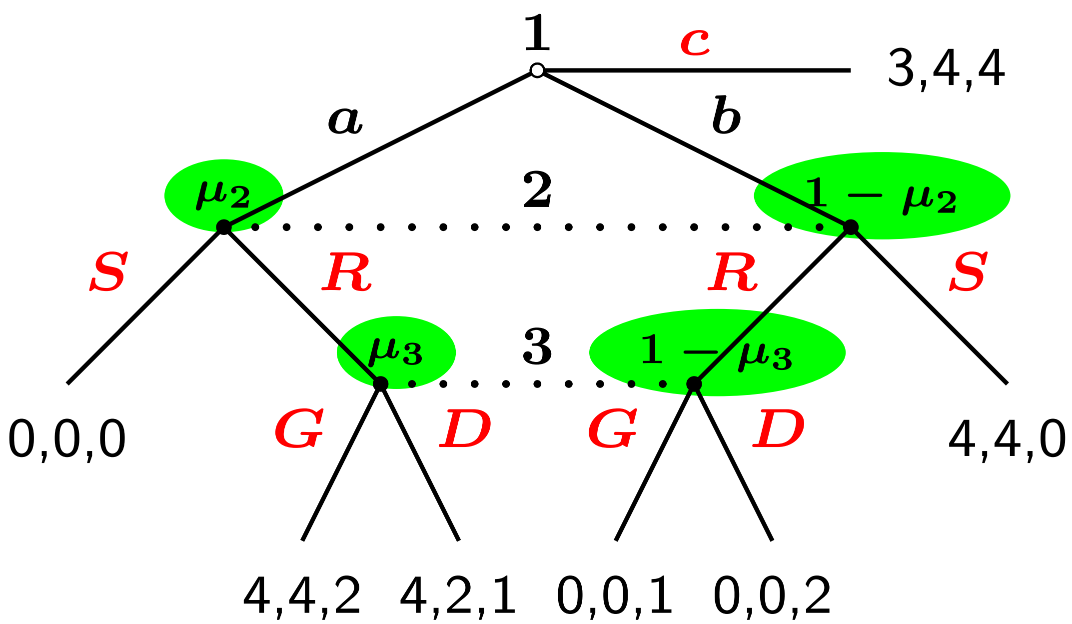

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Sequential rationality:

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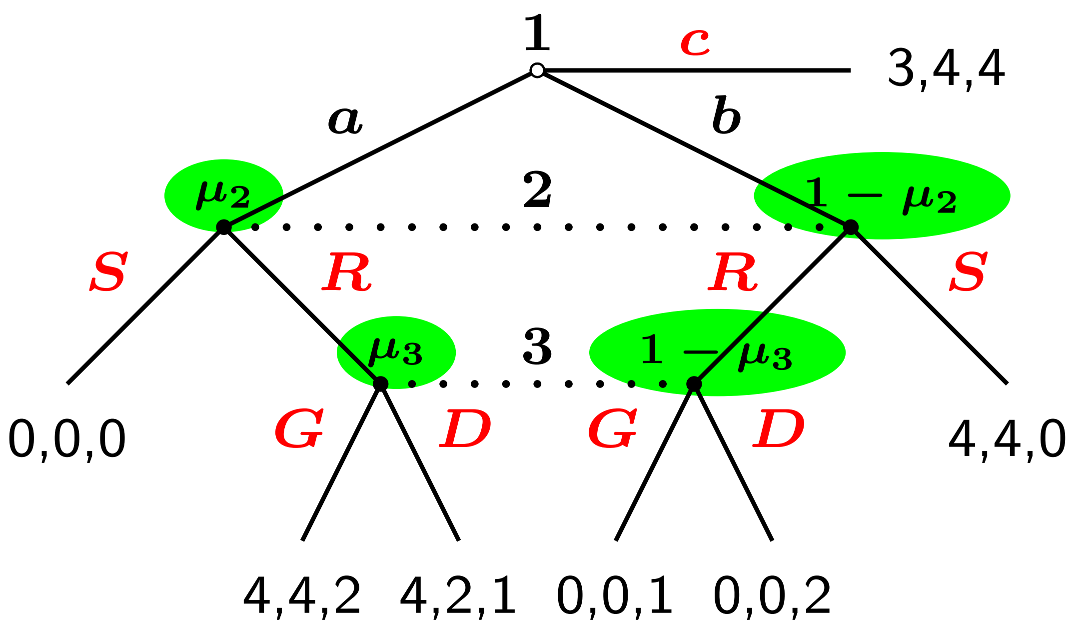

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Sequential rationality: Player 1 a and $b \longrightarrow 2$, $c \longrightarrow 3 \geq 2$ OK

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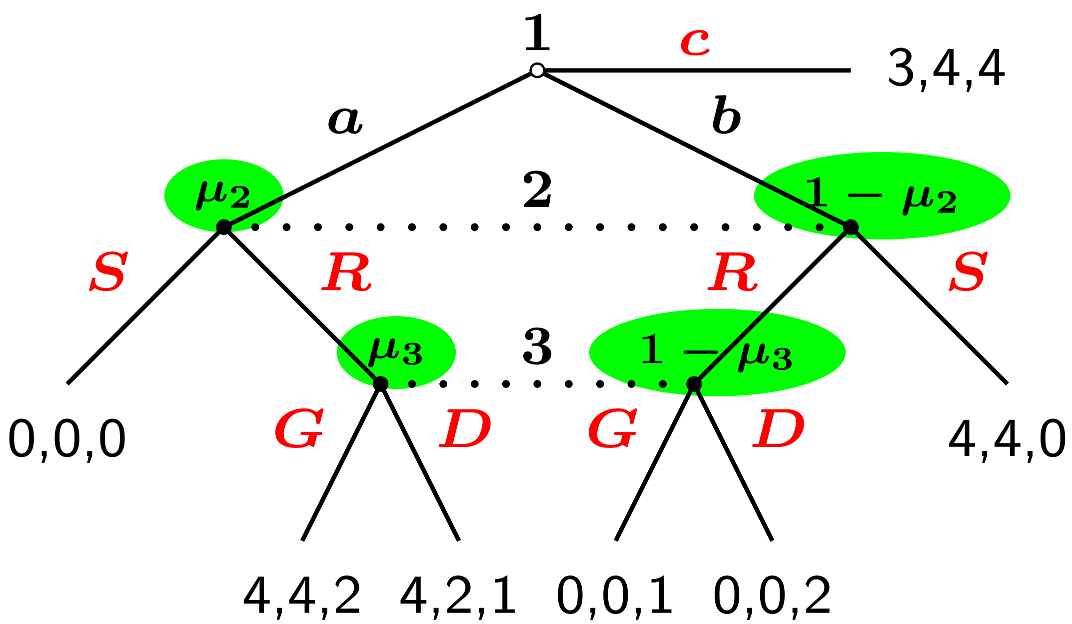

Consider the (SP)NE $(c, (\frac{1}{2}S + \frac{1}{2}R), (\frac{1}{2}G + \frac{1}{2}D))$

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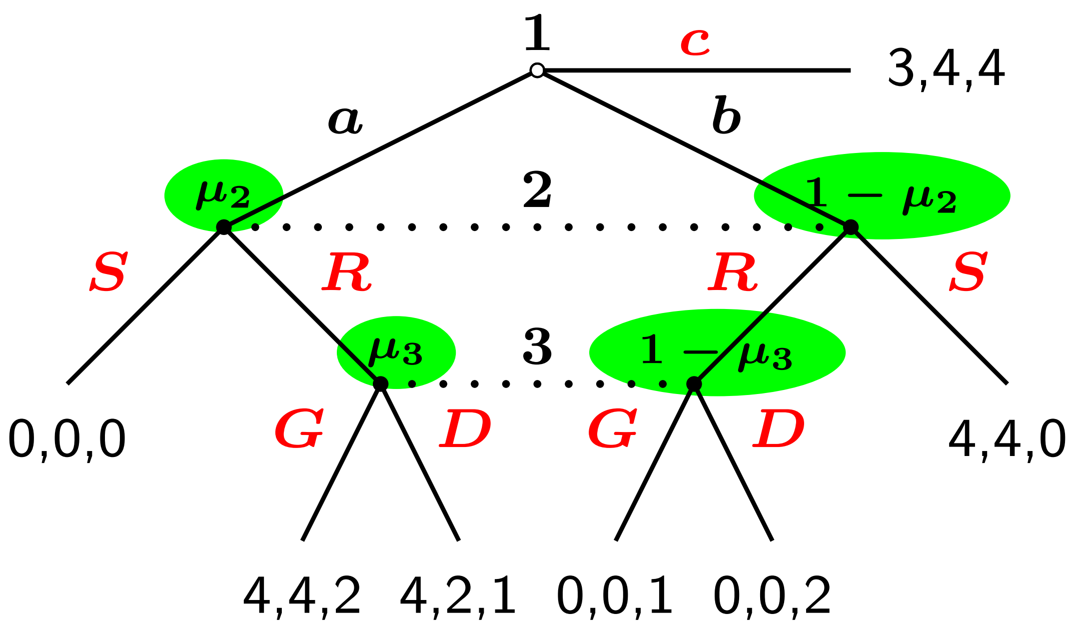

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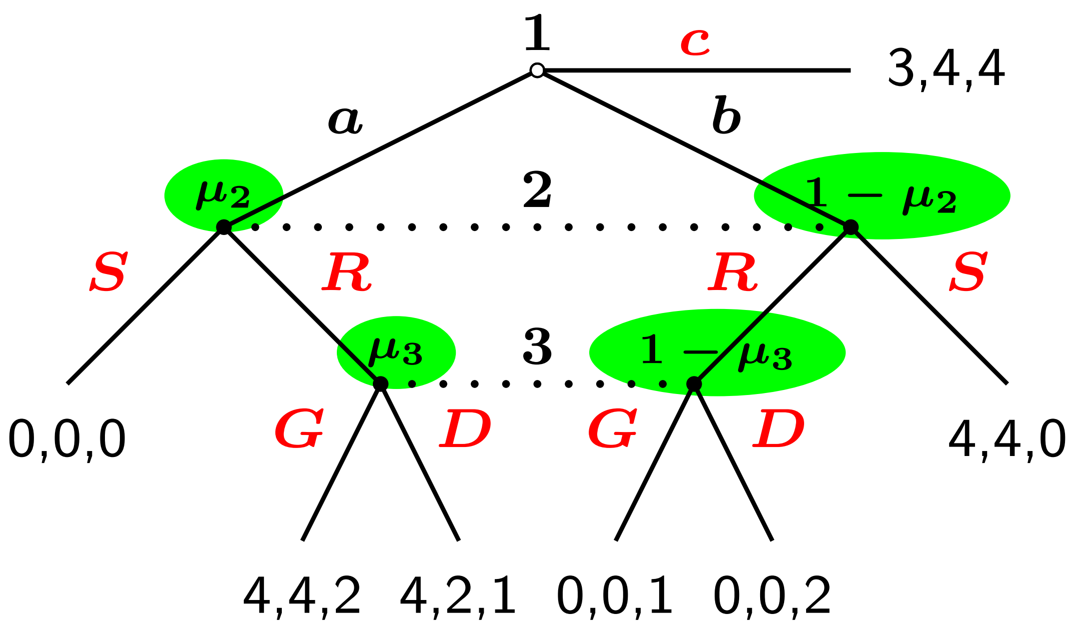

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But $\mu_2 = 4/7 \neq \mu_3 = 1/2$ is not strongly consistent: for every perturbed strategy profile $\tilde{\sigma}^k$ we have $\lim_\infty \tilde{\mu}_2^k = \lim_\infty \tilde{\mu}_3^k$

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More generally, we have:

$$\{SE\} \subseteq \{PBE\} \subseteq \{SPNE\} \subseteq \{NE\}$$

Proposition 3 *In games with perfect information the set of sequential equilibria (weak and strong) coincides with the set of SPNE*

Remark There exist stronger versions of perfect Bayesian equilibrium than those presented here, which apply to more specific dynamic games. For example, in some classes of multistage games with independent types, Fudenberg and Tirole (1991) define a perfect Bayesian equilibrium (without referring to perturbed strategies) which is equivalent to the (strong) sequential equilibrium

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A particularly simple class of dynamic games of incomplete information in which the simplest version of PBE and the strong SE coincide is the class of signaling games

Signaling Games

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➥ Strategies: $\sigma_1 : T \longrightarrow \Delta(M)$ and $\sigma_2 : M \longrightarrow \Delta(R)$

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Ex: If $M(t_1) = \{m_1, \overline{m}\}$ and $M(t_2) = \{m_2, \overline{m}\}$, then $m_i =$ certificate/proof that the sender's type is t_i

Binary case: 2 types / 2 messages / 2 responses

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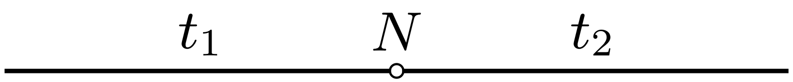

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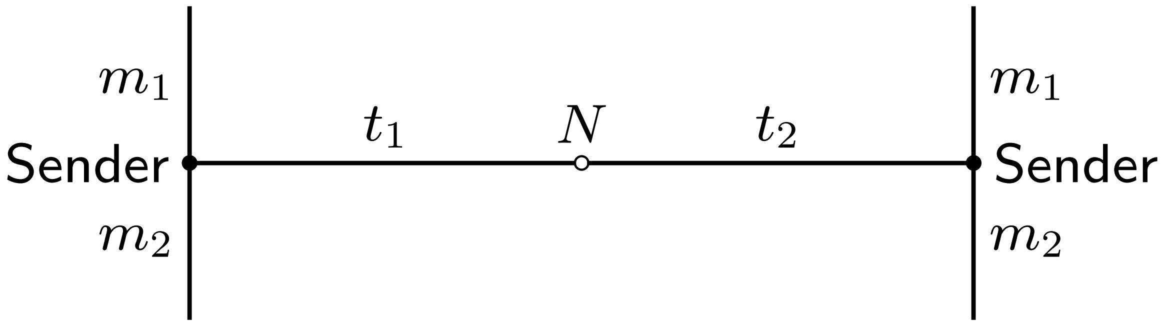

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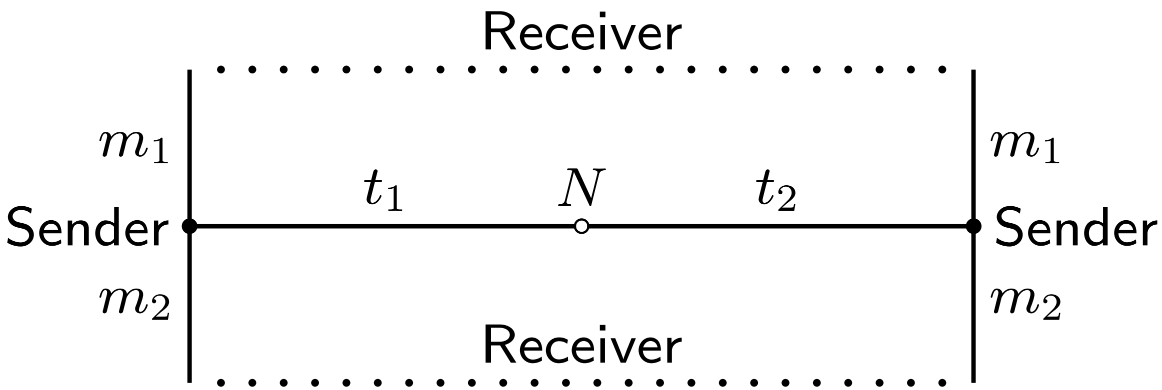

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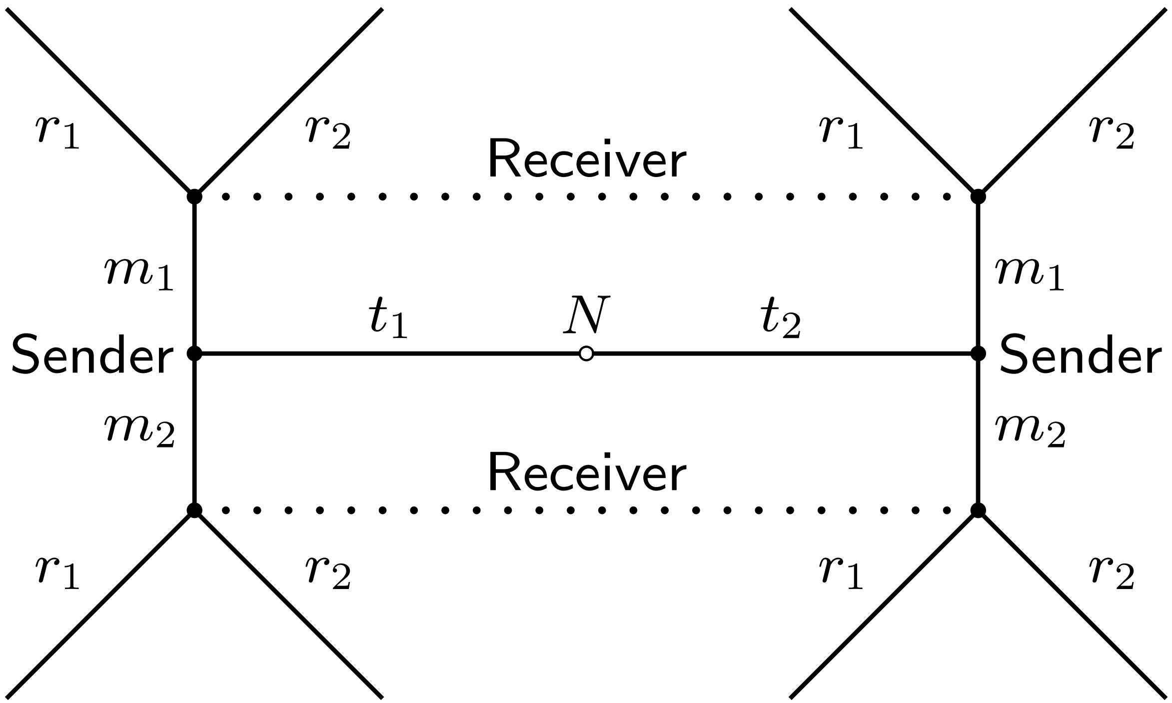

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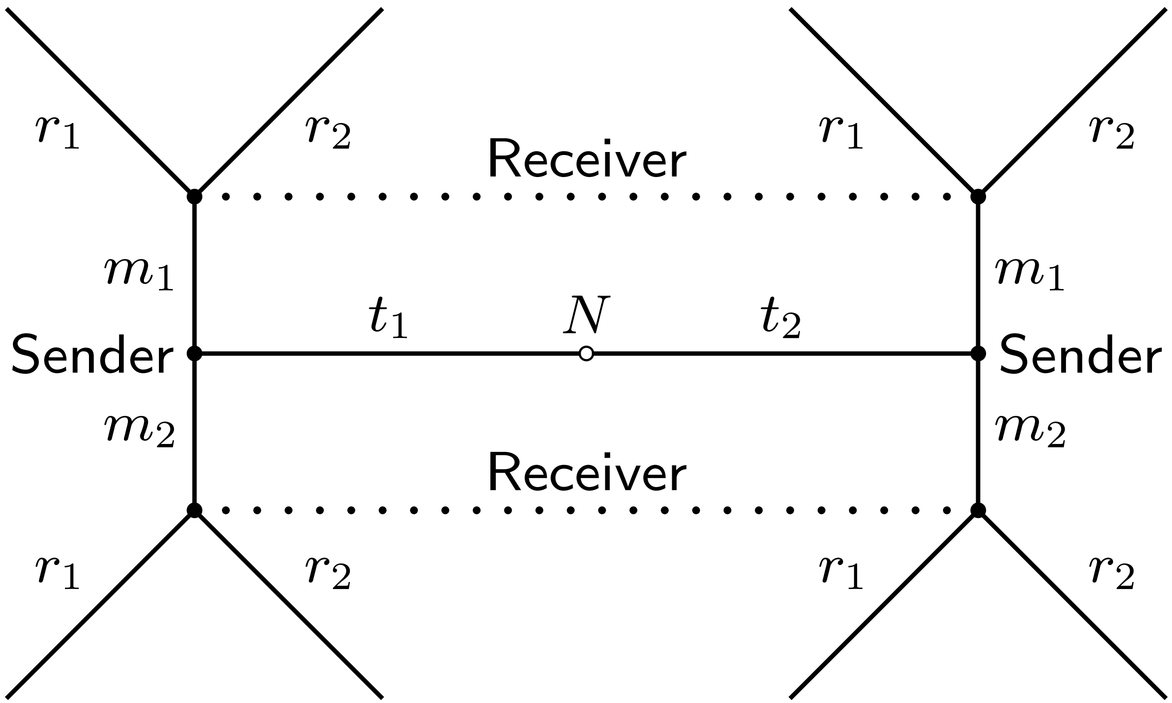

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✍ Difference with the definition of a Nash equilibrium of the signaling game?

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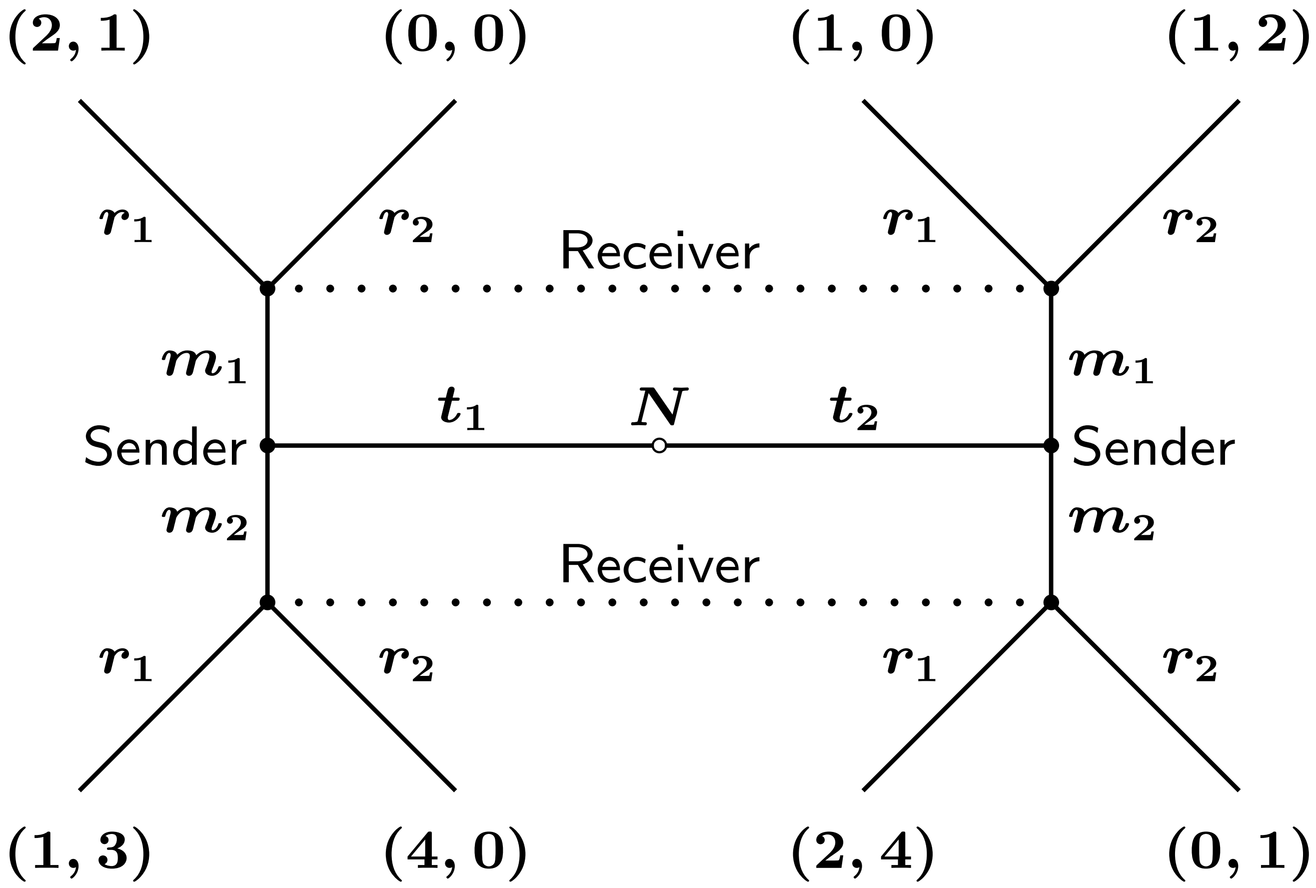

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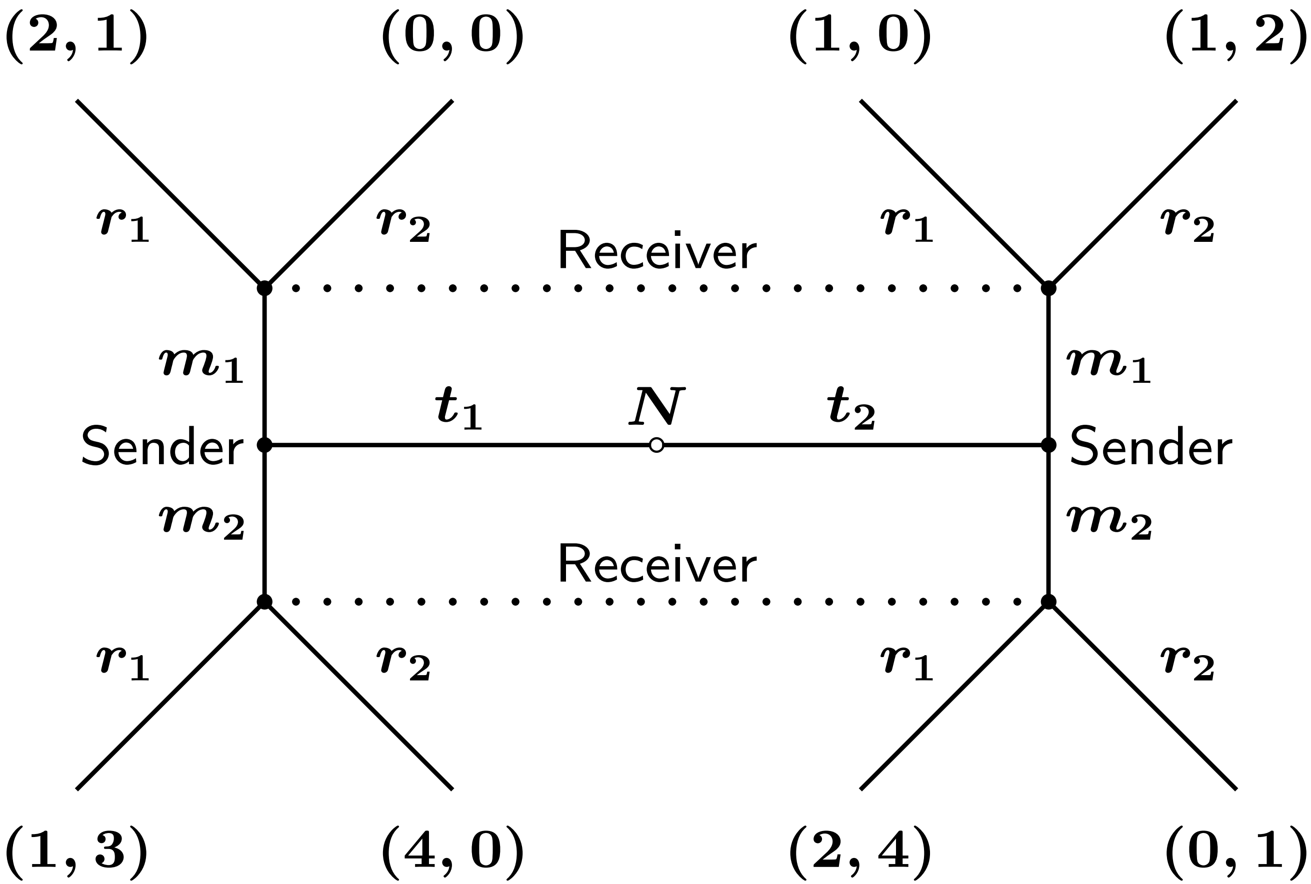

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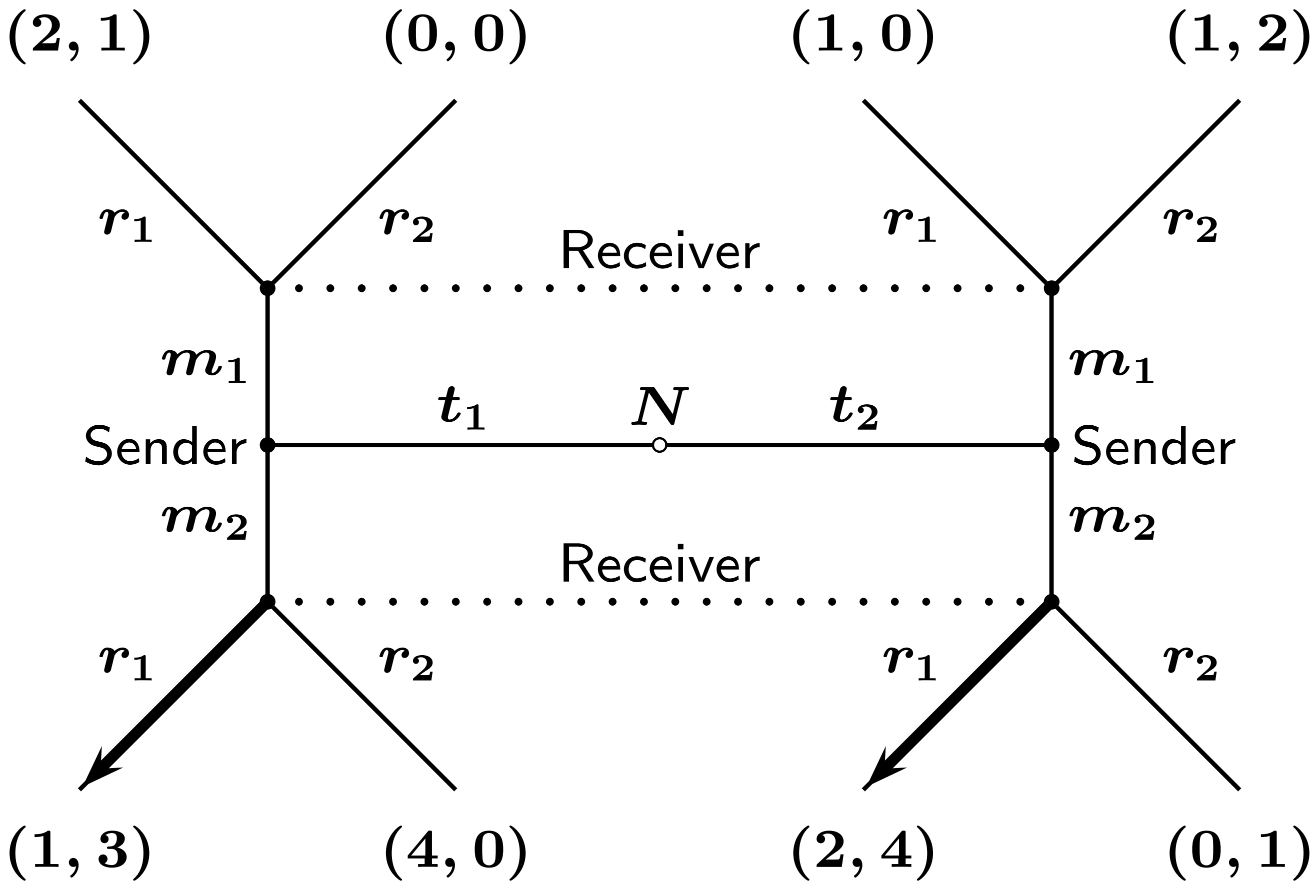

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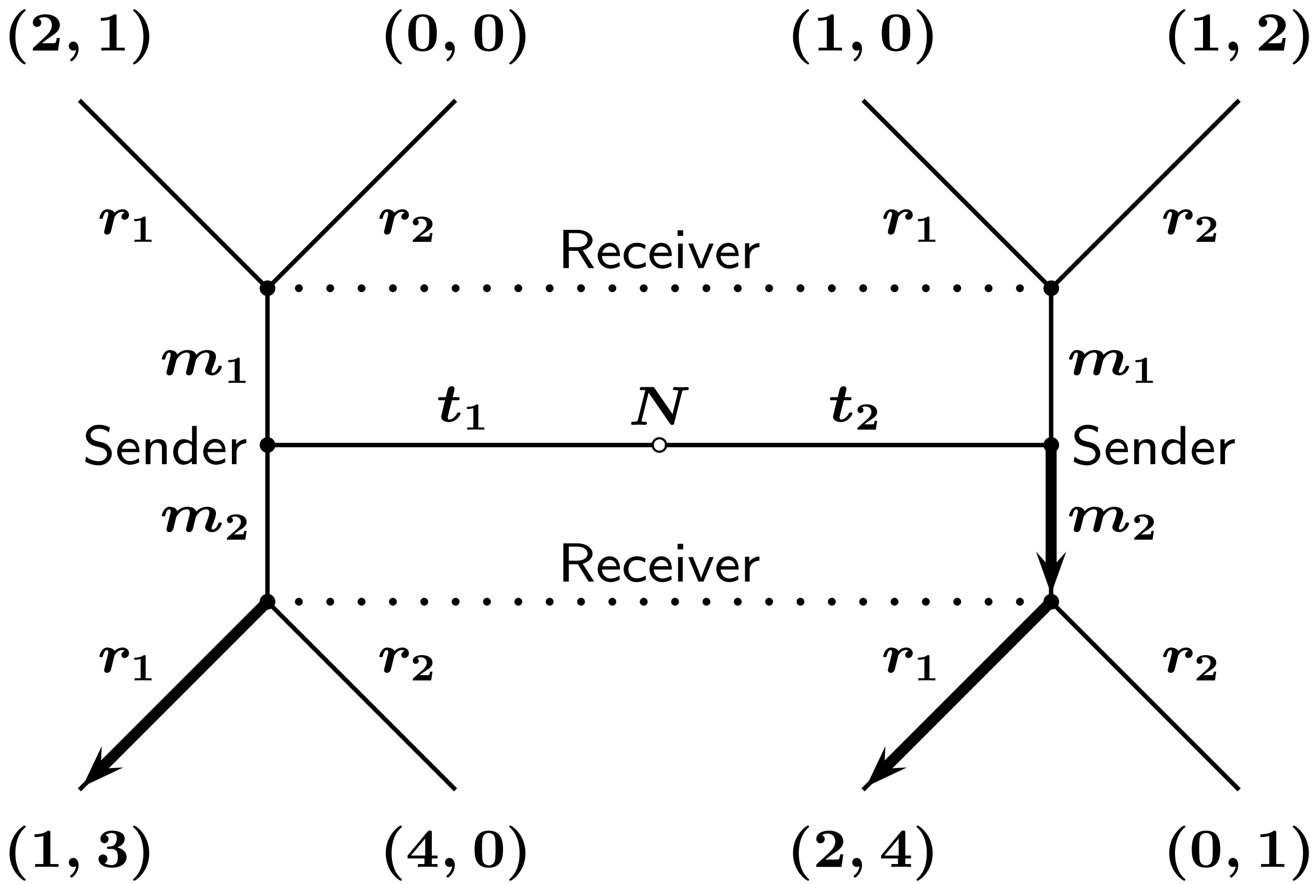

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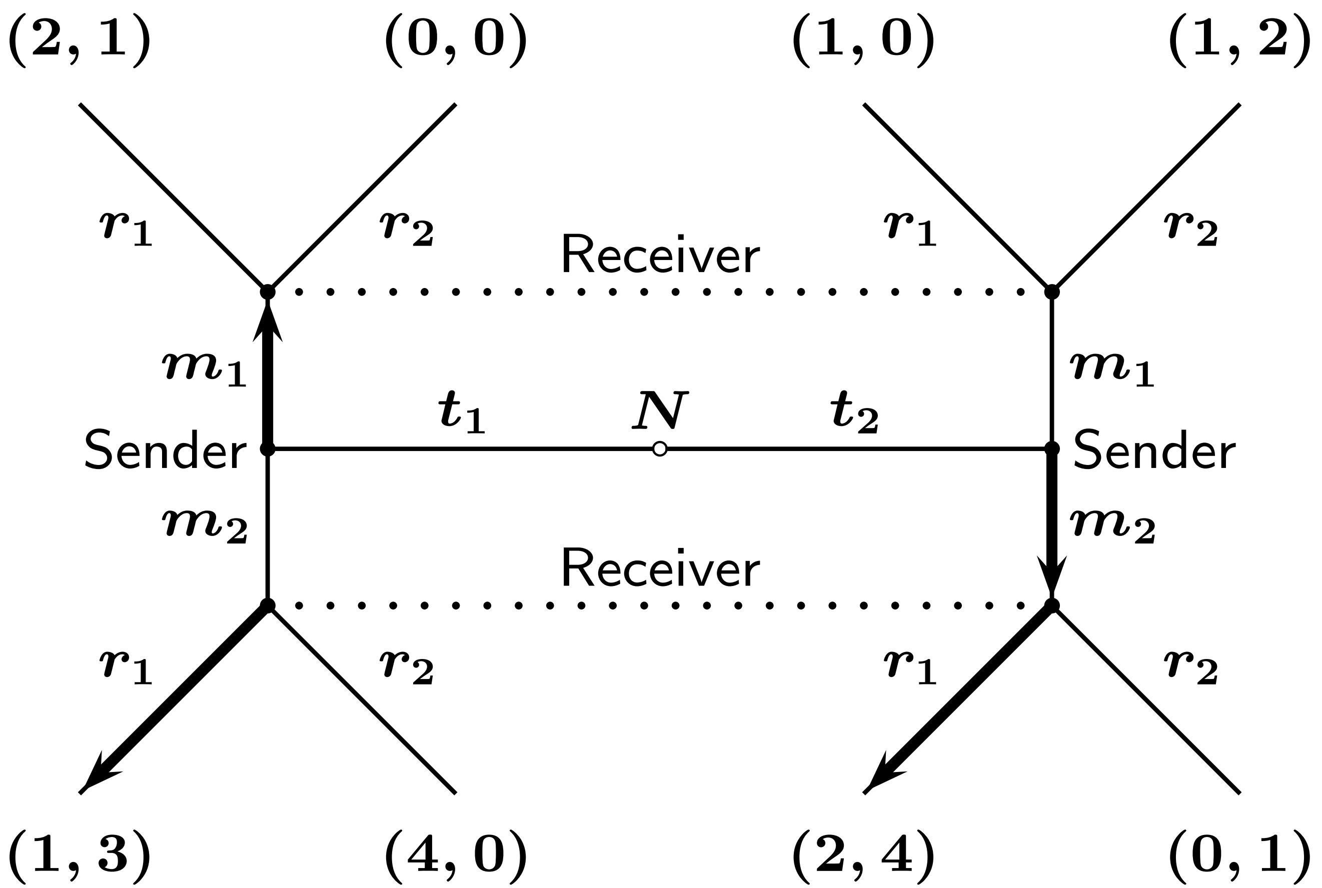

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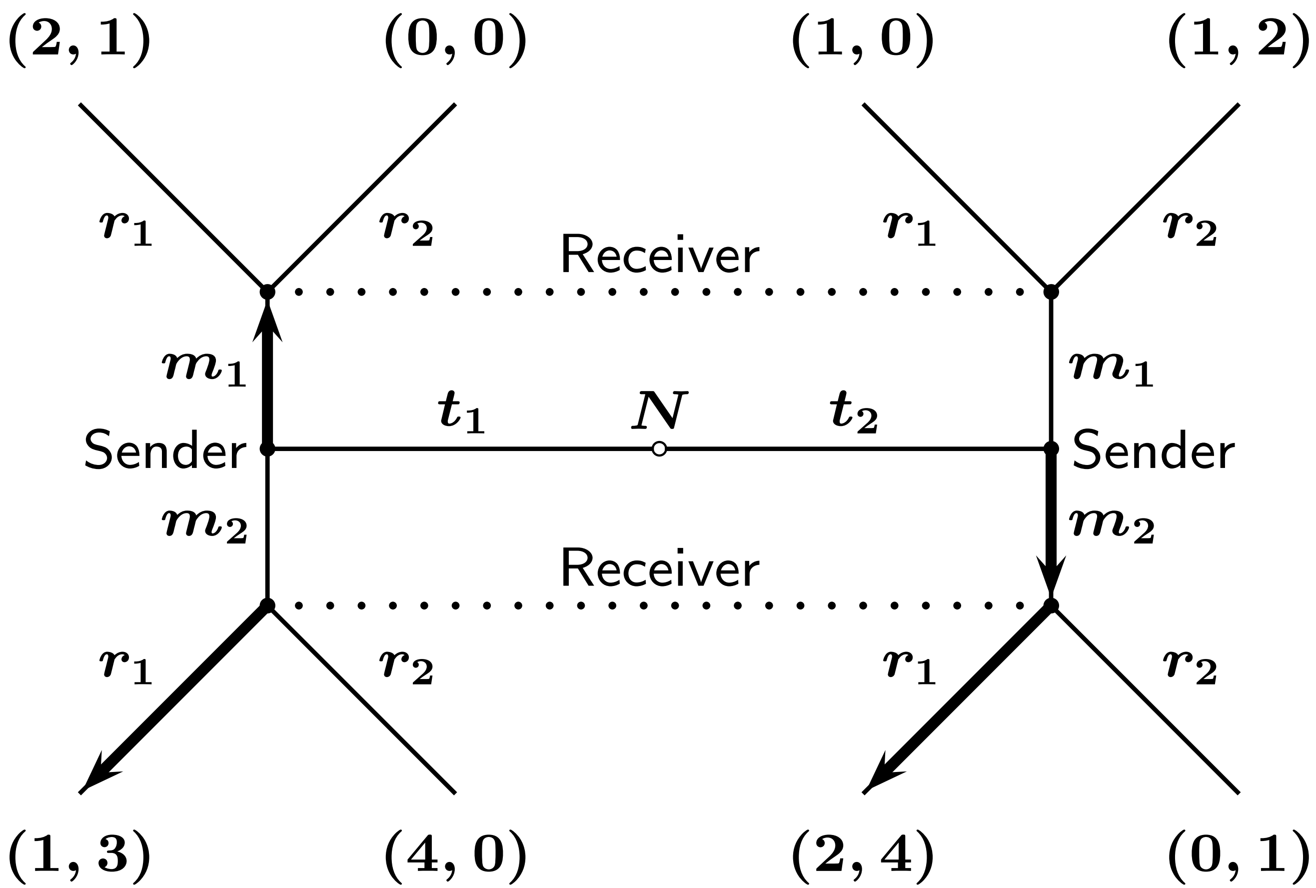

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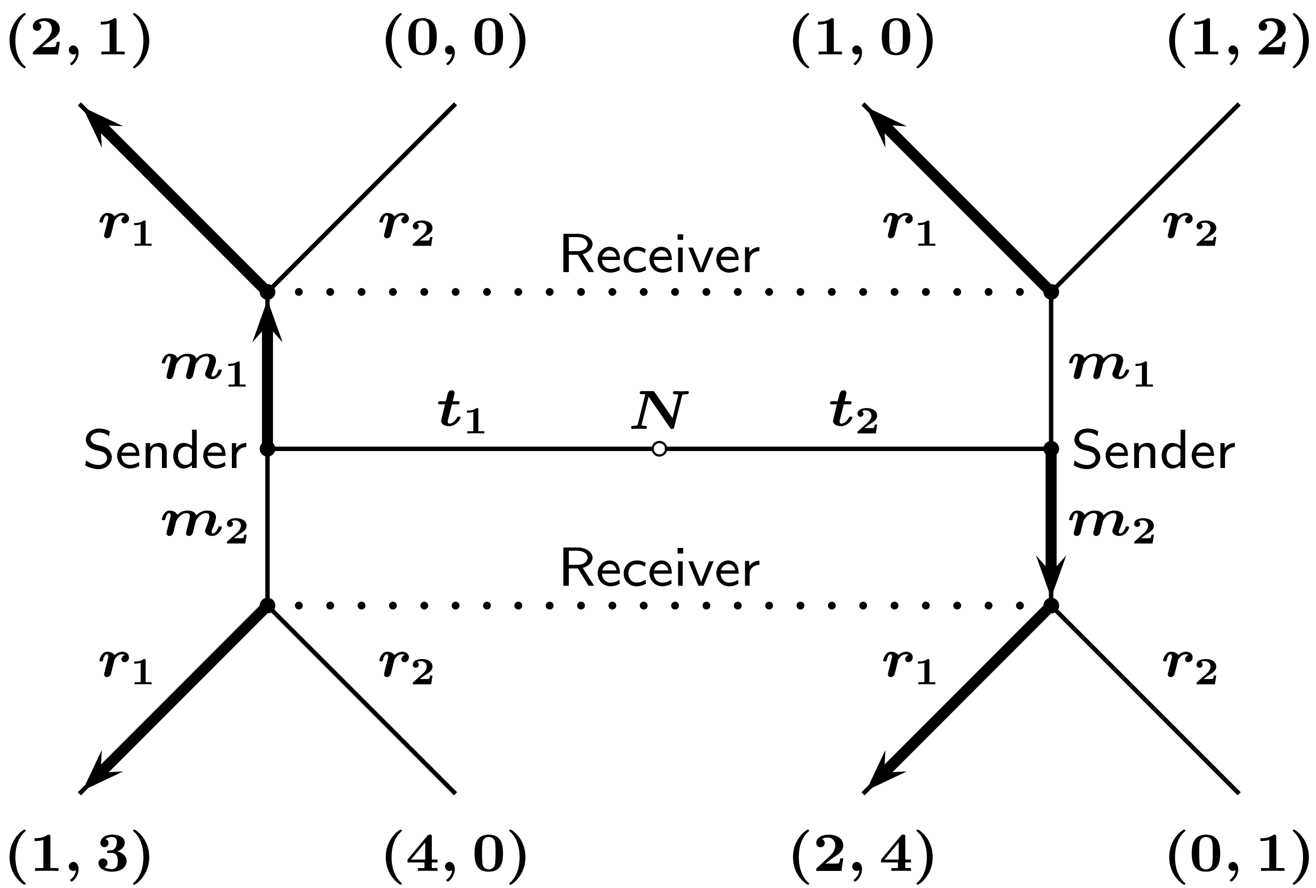

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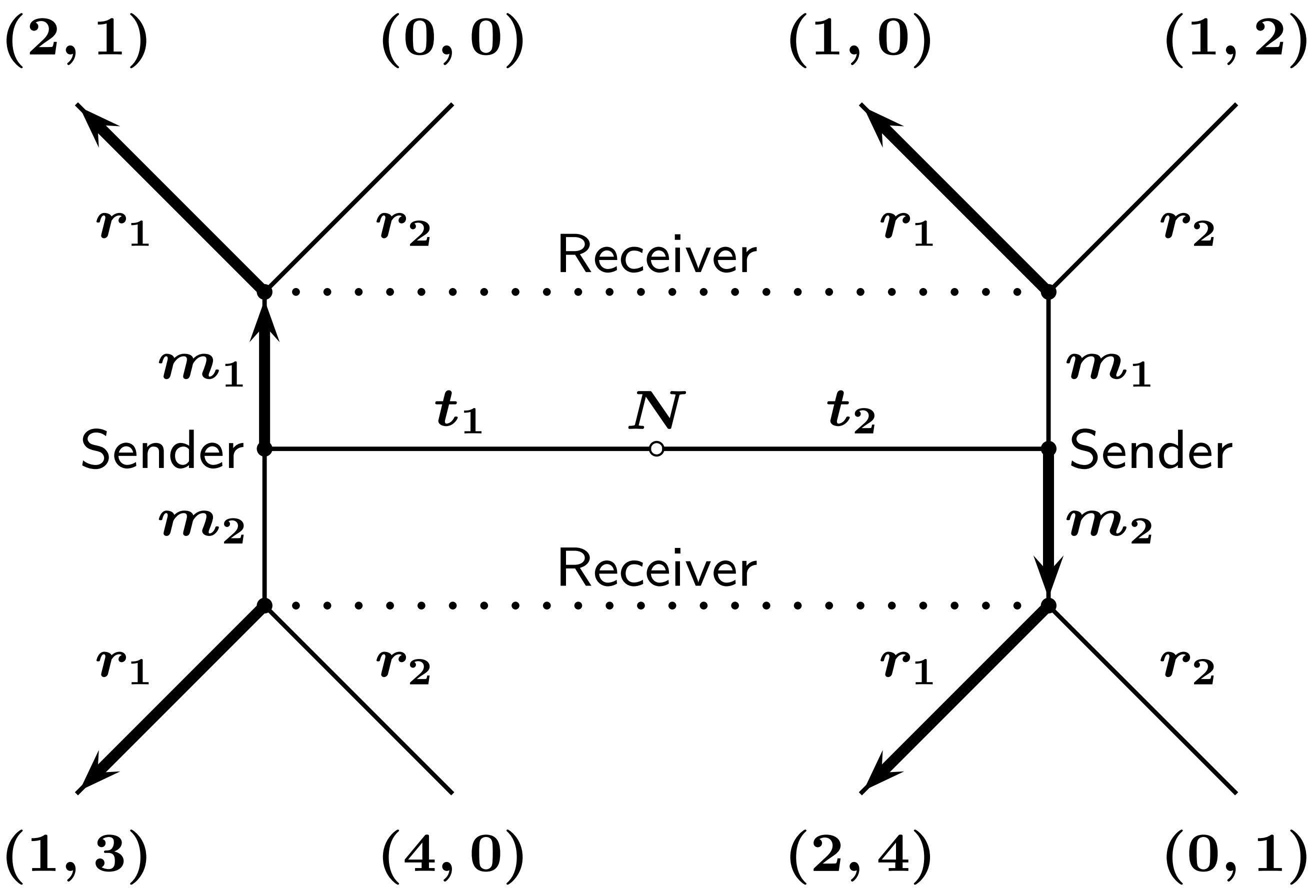

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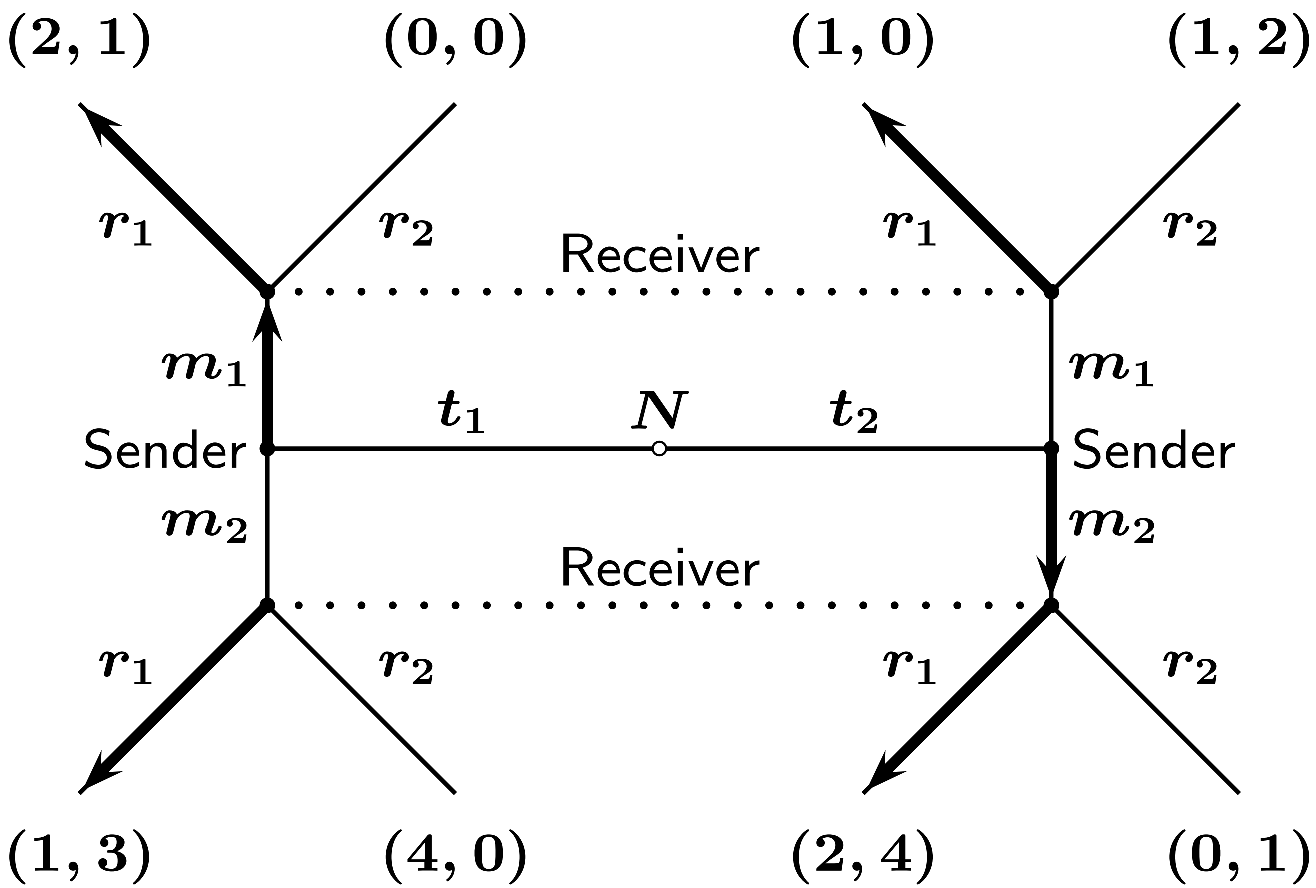

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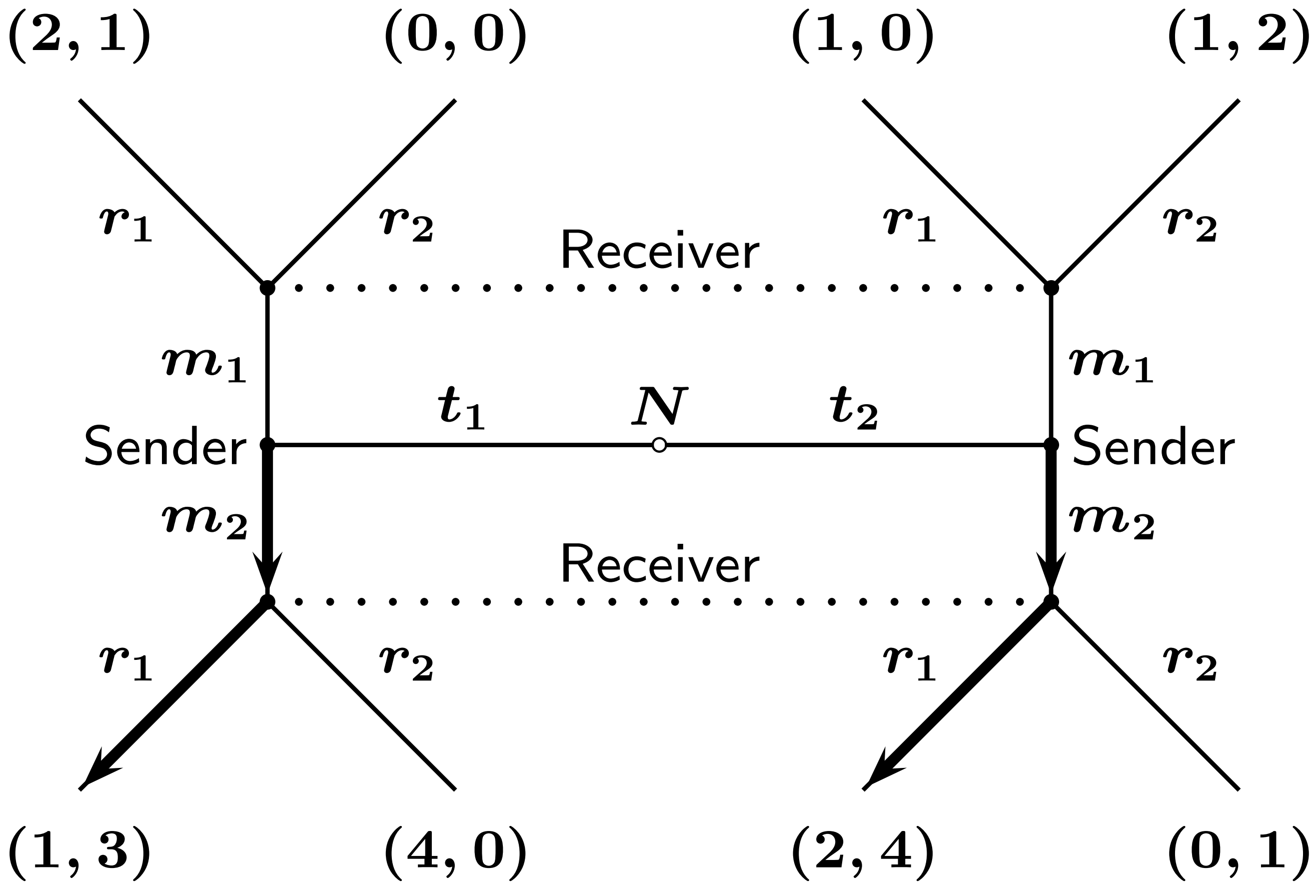

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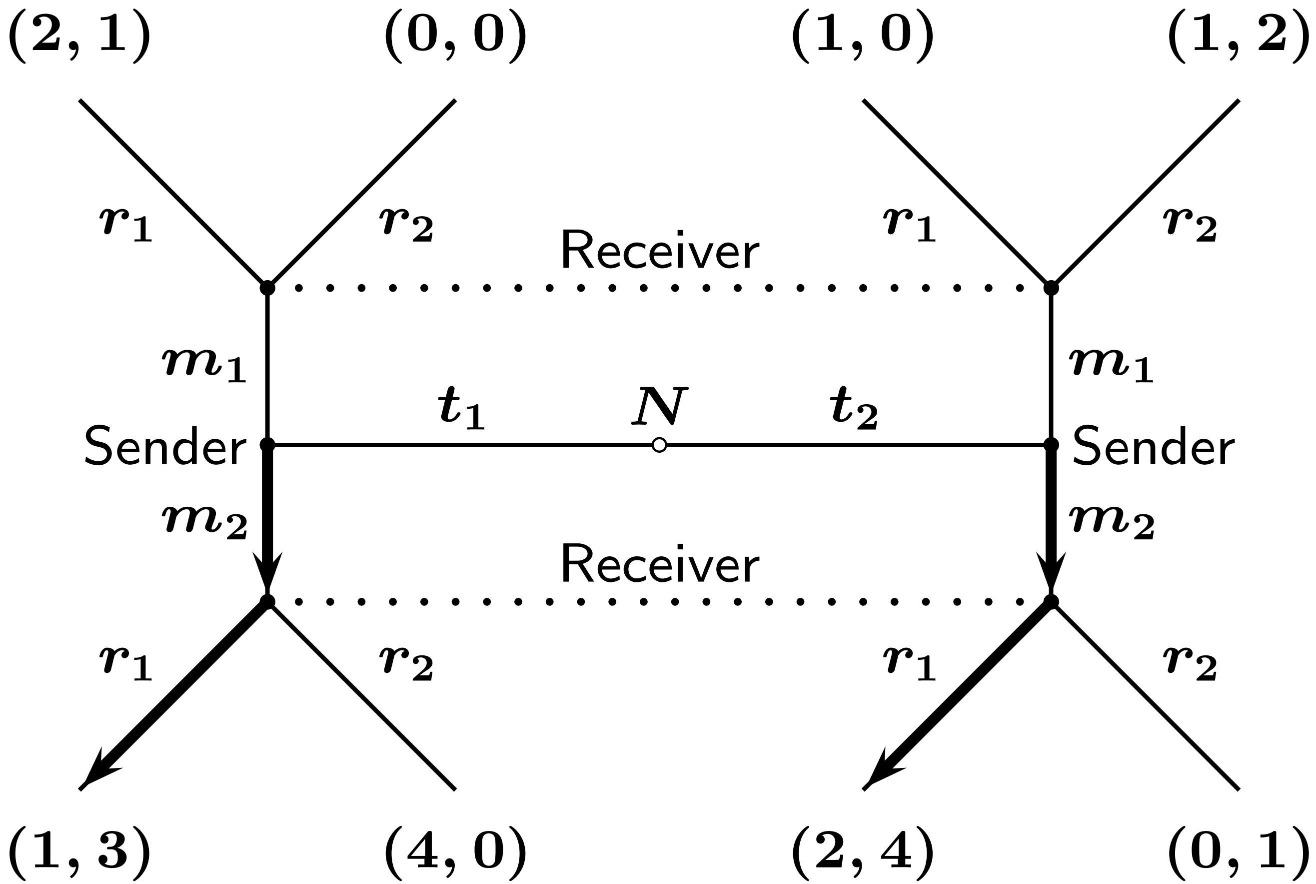

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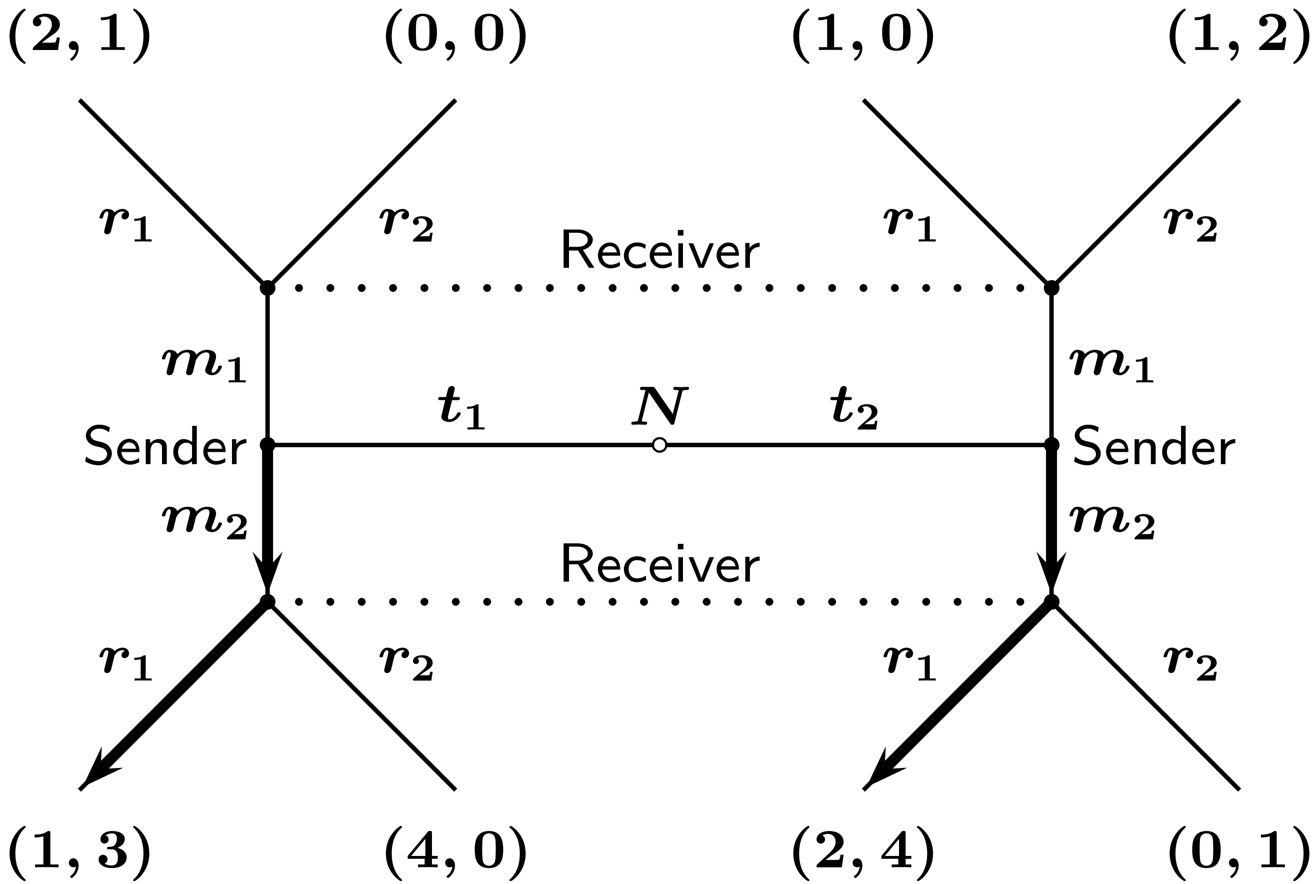

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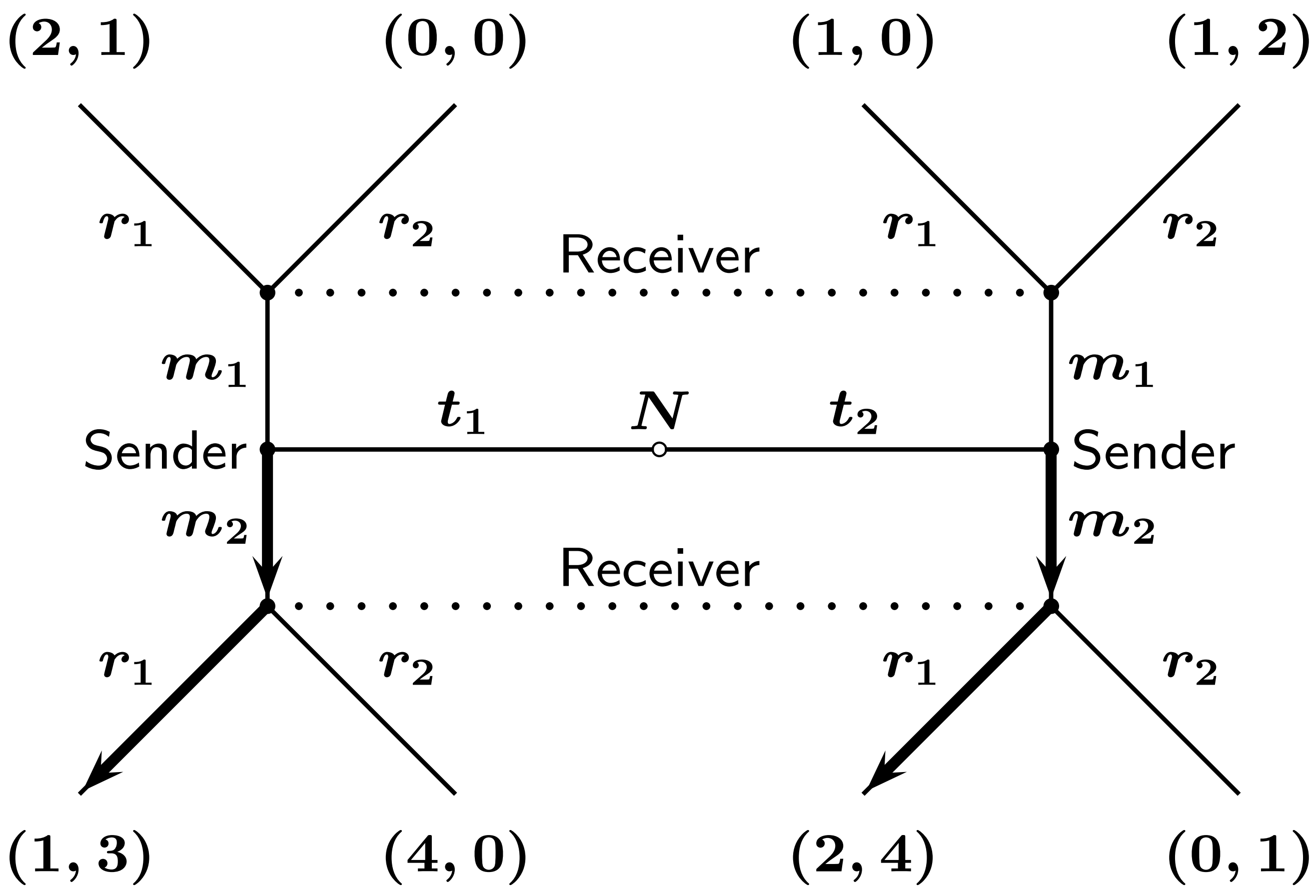

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⇒ $\mu_2(t_1 \mid m_1)$ should be smaller than 2/3

✍ Write the previous signaling game in normal form and show that the set of pure strategy Nash equilibria coincides with the set of pure strategy PBE

✍ Find a signaling game with a Nash equilibrium outcome which is not included in the set of PBE outcomes

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Idea: An agent's desutility (cost) for investing in a higher level of education is smaller for highly productive agents than for less productive agents

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$\Rightarrow$ A highly productive agent tends to invest in higher levels of education

Application: Spence's (1973) Model of Education

Signaling game (message = level of education) from a job candidate to employers (in competition) who don't know the ability (the productivity) of the candidate

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⇒ A highly productive agent tends to invest in higher levels of education

⇒ Potential employers understand this, and thus are willing to pay more workers with high levels of education, even if education has no direct impact on productivity

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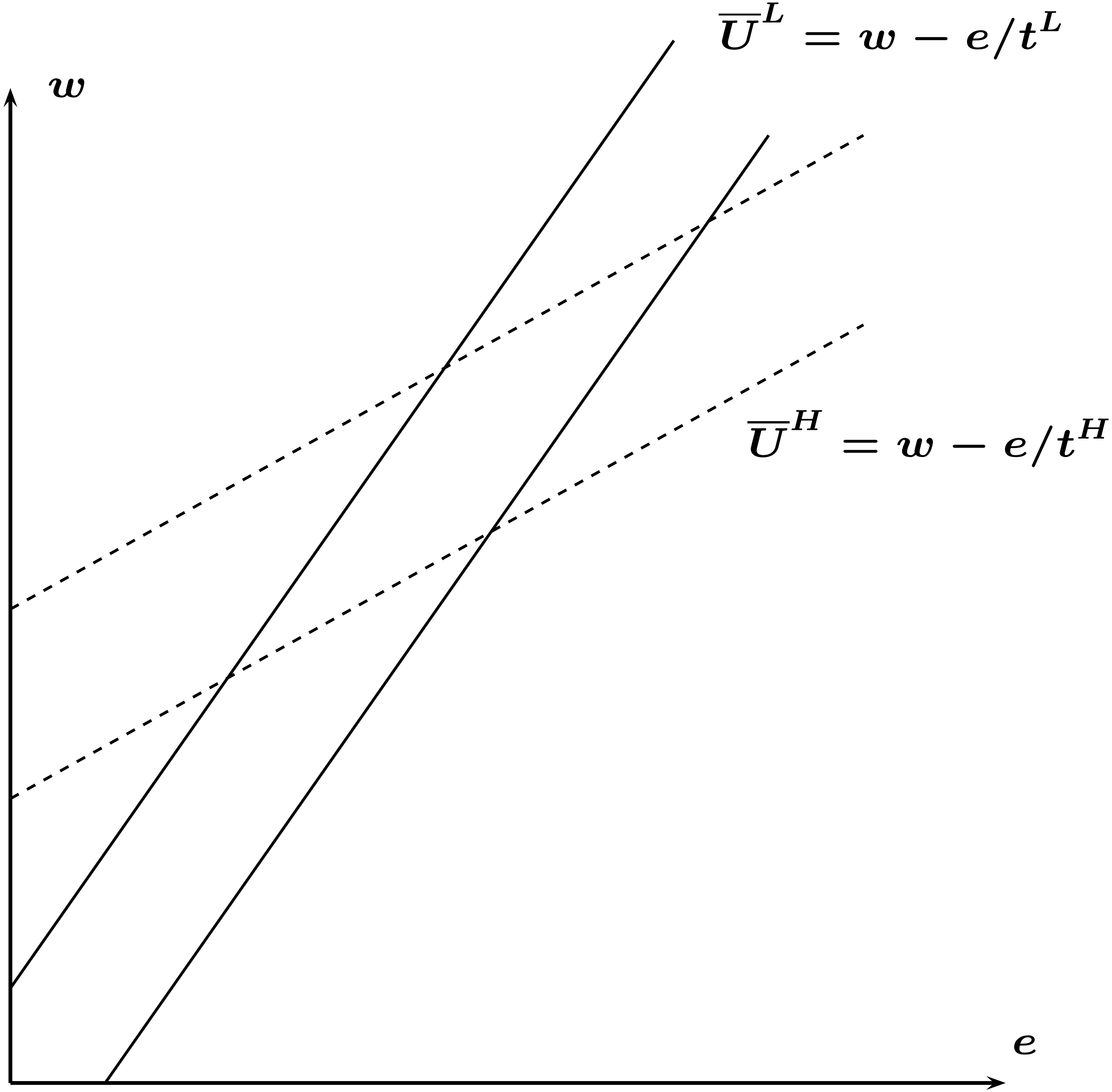
for example $c(t,e) = e/t$



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This "first best" solution (for the worker) is obviously not an equilibrium under asymmetric information since $e(t^L) = e(t^H) = 0$ does not reveal the worker's ability to the employer

- **Sequential rationality of the employer**: for every $e \geq 0$,

$$
\begin{aligned}
w(e) &= \mathbf{E}_{\mu}\big[y(\cdot, e) \mid e\big] \\
&= \mu(t^H \mid e)\, y(t^H, e) + \mu(t^L \mid e)\, y(t^L, e) \\
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$$e(t) \in \arg\max_e \; w(e) - c(t, e) = \arg\max_e \; \mu(t^H \mid e)\, (t^H - t^L) - e/t$$

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$$w(e) - c(t, e) = \mu(t^H \mid e)\,(t^H - t^L) + t^L - e/t$$

where $\mu(t^H \mid e) \in [0, 1]$ is the off the equilibrium belief of the employer. It can be chosen arbitrarily since Bayes' rule does not apply

The worker does not deviate if

$$p\,(t^H - t^L) - e_m/t \geq \mu(t^H \mid e)\,(t^H - t^L) - e/t \ \ \forall\, t,\ \forall\, e \geq 0$$

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Conclusion: Pooling PBE exist under the following conditions:

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✍ Show that there exist pooling Nash equilibria in which the worker chooses $e_m > t^L \, p \, (t^H - t^L)$ whatever his type. Explain why these Nash equilibria are not perfect Bayesian equilibria

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Conclusion: Separating PBE outcomes exist when $e(t^L) = 0$ and

$$t^L\,(t^H - t^L) \leq e(t^H) \leq t^H\,(t^H - t^L)$$

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Remark An more intuitive belief system, which is also consistent and support these equilibria, is $\mu(t^H \mid e) = \begin{cases} 1 & \text{if } e \geq e^H \\ 0 & \text{if } e < e^H \end{cases}$ so $w(e) = \begin{cases} t^H & \text{if } e \geq e^H \\ t^L & \text{if } e < e^H \end{cases}$

Remark Some stronger equilibrium refinements, based on forward induction (for example, the intuitive criterion of Cho and Kreps, 1987) allow to select as a unique equilibrium the most efficient separating equilibrium: $e(t^L) = 0$,
$e(t^H) = t^L\,(t^H - t^L)$

Idea of the refinement: If e is a strictly dominated action for the sender of type t, but not for the sender of type t', then $\mu(t \mid e) = 0$

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Other applications:

- Advertising (Milgrom and Roberts, 1986): A firm selling a high quality product can signal this quality with expensive advertising if a firm with a low quality product is not able to cover these advertising costs given its future profits

Remark Some stronger equilibrium refinements, based on forward induction (for example, the intuitive criterion of Cho and Kreps, 1987) allow to select as a unique equilibrium the most efficient separating equilibrium: $e(t^L) = 0$, $e(t^H) = t^L\,(t^H - t^L)$

Idea of the refinement: If e is a strictly dominated action for the sender of type t, but not for the sender of type t', then $\mu(t \mid e) = 0$

Other applications:

- Advertising (Milgrom and Roberts, 1986): A firm selling a high quality product can signal this quality with expensive advertising if a firm with a low quality product is not able to cover these advertising costs given its future profits
- Insurance (Rothschild and Stiglitz, 1976; Wilson, 1977): A risk-averse driver will purchase a lower cost, partial insurance contract, leaving the riskier driver to pay a high rate for full insurance

- Bargaining: The magnitude of the offer of a firm to a union may reveal its profitability if firms with low profits are better able to make low wage offers (because the threat of a strike is less costly to a firm with low profits)

- Bargaining: The magnitude of the offer of a firm to a union may reveal its profitability if firms with low profits are better able to make low wage offers (because the threat of a strike is less costly to a firm with low profits)
- Evolutionary biology (Zahavi, 1975; Grafen, 1990: handicap principle): a peacock's tail may be a signal used by prospective mates in order to estimate the individual's overall condition and/or genetic quality. Indeed, only the strongest individuals should be able to survive to predators with such an handicap. The same principle can explain why gazelles jump up and down when they see a lion, . . .

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